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Rostock, 18.02.2025

Exam „Introduction to Numerical Mathematics“

Problem 1 (4 points):

Compute approximations to the solution of the system of linear equations $Ax = b$ with

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}.$$

(a) Calculate one step using Gauss-Seidel's method starting with $x^{(0)} = (1,1,1)^T$.

(b) Calculate one step using Jacobi's method starting with the same $x^{(0)}$.

(c) Compute the error propagation matrix of Jacobi's method for this certain problem.

Problem 2 (5 points):

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

(a) Compute two steps with the power method for the starting vector $x = (0,1,0)^T$ and give the Rayleigh-coefficient.

(b) Compute one step with the inverse power method for the starting vector $x = \sqrt{0.5}(1,0,1)^T$ and give the Rayleigh-coefficient. Use

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 & -4 \\ 2 & -1 & 2 \\ -4 & 2 & 1 \end{pmatrix}.$$

(c) Apply the inverse power method with shift and use $\mu = 2$. Compute one step and the Rayleigh-coefficient. Start with $x = (1,0,0)^T$ and use

$$(A - 2I)^{-1} = \frac{1}{9}\begin{pmatrix} -5 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & -5 \end{pmatrix}.$$

Turn the page.

Problem 3 (5 points):

Compute the value

$$s = \lim_{k\to\infty} \underbrace{\frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}}_{k \text{ fractions}}.$$

Proceed as follows for this purpose:

(a) Specify a step function φ with s being a fixed point of φ.

(b) Specify an interval $I = [\alpha, \beta]$ with $\alpha < \beta$ so that the requirements of Banach's fixed point theorem are met. Check them.

(c) Compute 3 iterations with a valid $x_0 \in I$.

(d) What is the exact value?

Problem 4 (4 points):

To indicate the efficiency η of a machine depending on the temperature T a manufacturer specifies three pairs of values as follows

T [in °C]	-7	2	10
η	2	3.5	4.5

(a) What efficiency can be assumed for $T = 5$°C? For this, determine the interpolation polynomial $p(x)$ for these points (use a method of your choice) and compute $p(5)$.

(b) Tests suggest that $\eta(0) = 3$. Compute the interpolation polynomial regarding this additional information using Newton's scheme.

(c) Omit the assumption from (b) and assume instead that the machine has its maximal efficiency for $T = 10$°C. Compute the interpolation polynomial taking this information into account.

Problem 5 (6 points):

Consider the function

$$f : \mathbb{R}_+ \to \mathbb{R}, \quad f(x) = \frac{1}{x^2}.$$

(a) Use composite Simpson's rule S_4 with $n = 4$ subintervals to approximate $I = \int_1^5 f(x)\,\mathrm{d}x$.

(b) How large must n be in order to be sure that the error is smaller than $\varepsilon = 10^{-4}$?
HINT: For Simpson's rule holds

$$\left| \int_a^b f(x)\,\mathrm{d}x - S_n \right| \leq \frac{(b-a)h^4}{180} \max_{x\in[a,b]} |f^{(4)}(x)|.$$

(c) Approximate $\int_1^\infty f(x)\,\mathrm{d}x$ and use the coordinate transformation $x(z) = \tan(z)$ with

$$x' = \frac{\mathrm{d}x}{\mathrm{d}z} = \frac{1}{\cos(z)^2} \quad \Rightarrow \quad \mathrm{d}x = \frac{1}{\cos(z)^2}\,\mathrm{d}z$$

and composite Gaussian rule Q_1 with 2 nodes and 2 intervals.

PLEASE SCROLL ON.

Problem 6 (6 points):

Consider the initial value problem

$$\begin{aligned} y'(x) &= -5y(x), \quad x \in [0,5], \\ y(0) &= 1. \end{aligned}$$

(a) Apply two steps of Euler's method using $h = 0.5$.

(b) Apply one step of the classical Runge-Kutta method using $h = 1$.

(c) For this ODE and Euler's method with $h = 0.5$ it is possible to derive an explicit formula for Y_{k+1} depending only on k. Determine this formula and compute Y_{10} without computing $Y_2, \ldots, Y_9$.

(d) Show that the application of the method

$$Y_{k+1} = Y_k + hf\big(x_k + h/2, (Y_k + Y_{k+1})/2\big)$$

to the problem results in

$$Y_n = Y_0 \cdot \left(\frac{1 - 2.5h}{1 + 2.5h}\right)^n.$$

Compute Y_{10} for $h = 0.5$.