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Exam "Introduction to Numerical Mathematics"

Problem 1 (5 points):

We search for the intersection x^* of the functions

$$f(x) = 1/x, \quad g(x) = \ln(x).$$

(a) Show that the function $\varphi(x) = \exp(1/x)$ fulfills the conditions of Banach's fix point theorem for the interval $I = [1.5, 2]$.

(b) Compute 3 iterations with $x_0 = 1.75$.

(c) Apply an a posteriori error estimation to compute an upper bound for $|x_3 - x^*|$.

(d) Approximate x^* also by Newton's method. Use an appropriate function $h(x)$ with $h(x^*) = 0$ and carry out one iteration for $x_0 = 1.75$.

Problem 2 (6 points):

(a) Compute the interpolation polynomial $p(x)$ through the points

x	-2	0	3
y	-3	1	4

that additionally fulfills $p'(3) = 2$.

(b) Derive a finite difference to approximate f''' for a function $f : \mathbb{R} \to \mathbb{R}$ that is three times continuously differentiable. Use $f(x-h)$, $f(x)$, $f(x+h)$, and $f(x+2h)$ for a small $h > 0$.

Apply the formula to $f(x) = x^3$ for $x_0 = 1$ using $h = 0.1$.

Problem 3 (5 points):

(a) Approximate $I = \int_0^3 f(x)\,\mathrm{d}x$ for $f = \exp(-x^2)$ by composite Simpson's rule S_4 with $n = 4$ subintervals.

(b) What is the maximal number of subintervals required so that the difference between the approximation and the exact value is smaller than $\varepsilon = 10^{-4}$?

Hints: It holds $\max_{x\in[0,3]} |f^{(4)}(x)| = 12$ and for Simpson's rule

$$\left| \int_a^b f(x)\,\mathrm{d}x - S_n \right| \le \frac{(b-a)h^4}{180} \max_{x\in[a,b]} |f^{(4)}(x)|.$$

(c) Apply composite Gaussian rule with 2 nodes and 2 intervals to approximate I from (a).

Turn the page.

Problem 4 (5 points):

Consider the initial value problem

$$\begin{aligned} y'(x) &= -2xy(x), \quad x \in [1,2], \\ y(1) &= 1. \end{aligned}$$

(a) Compute three steps of the explicit Euler method using $h = 1/3$.

(b) Compute one step using $h = 1$ of the method with the Butcher-tableau

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}.$$

(c) Apply implicit Euler method and compute two steps with $h = 2$.

Problem 5 (5 points):

Consider the system of linear equations $Ax = b$ with

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

(a) Compute two steps using Jacobi's method and the starting vector $x^{(0)} = (0, -1, 1)^T$.

(b) Compute one step using Gauss-Seidel's method starting with the same $x^{(0)}$ as in (a).

(c) Compute the error propagation matrix E of Jacobi's method for this certain problem.

(d) For the error propagation matrix of Jacobi's method holds $\|E\|_2 = \sqrt{0.5}$. Apply an a priori estimation how many steps are necessary to have $\|x^{(n)} - x^*\|_2 < 10^{-4}$ for $x^{(0)}$ and $x^{(1)}$ from (a).

Problem 6 (4 points):

Consider the symmetric matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

(a) Use Gerschgorin-circles to determine intervals for the eigenvalues of A.

(b) Calculate two steps with the power method and compute the Rayleigh-coefficients. Use the starting vector $x = (0, \sqrt{0.5}, \sqrt{0.5}, 0)^T$.