University of Rostock
Institute of Mathematics
PD Dr. M. Sawall

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3. Tutorial on the lecture „Introduction to Numerical Mathematics“

Problem 10:
The function $f(x) = \cos^2 x$ has exactly one fixed-point. Specify a largest possible interval (a, b) such that the conditions of Banach's fixed-point theorem are satisfied for any closed $I \subset (a, b)$.

Consider once again problem 30. Let $x_0 = 0.6$. Give an a-priori estimation for the error after k steps as accurate as possible. Determine for this, starting from x_0, a Lipschitz constant L as small as possible so that a contraction is present, and use L for the estimation. After how many steps is x_k certainly not further away from the fixed-point than $\varepsilon = 10^{-2}$?

Compute three steps and an a-posteriori error estimation for the last two iterations.

Problem 11:
Use Newton's method on the example of calculating $\sqrt{2}$ as solution of $f(x) = x^2 - 2 = 0$. Use as starting values $x_0 = 2$ and $x_0 = 100$ and as stopping criterion the error of the residual norm dropping below 10^{-6}.

Problem 12:
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with the fixed point $\bar{x}$ and $|\varphi'(\bar{x})| \neq 1$. Consider the fixed-point forms

$$\begin{array}{lll} 1. & x_{k+1} := \varphi(x_k) & \\ 2. & x_{k+1} := \varphi^{-1}(x_k) & \end{array} \qquad k = 0, 1, 2, \ldots .$$

Show that at least one of the fixed-point forms converge to the fixed point $\bar{x}$.

The tasks are intended both for processing in the seminars and for independent practice. Especially the 90 minutes of an exercise are sometimes not sufficient to discuss and work on all tasks.