University of Rostock
Institute of Mathematics
PD Dr. M. Sawall

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9. Tutorial on the lecture „Introduction to Numerical Mathematics“

Problem 32:
Consider Crank-Nicolson's method

$$Y_{k+1} = Y_k + \frac{h}{2}\big(f(x_k, Y_k) + f(x_{k+1}, Y_{k+1})\big).$$

(a) Show that the method has at least order 2.

HINT: To this end use $f(x+h, y(x+h)) = f(x+h, y(x) + hy'(x) + \mathcal{O}(h^2))$.

(b) Determine the stability domain of the method (model probelm $y' = \lambda y$).

Problem 33:
Consider the explicit 2-stage Runge-Kutta method Φ_1 with the Butcher tableau

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

and the implicit 1-stage Runge-Kutta method Φ_2 with the iteration

$$Y_{k+1} = Y_k + hf\big(x_k + \frac{1}{2}h, \frac{1}{2}Y_k + \frac{1}{2}Y_{k+1}\big).$$

(a) Compute two steps of Φ_1 for the initial value problem $y'(x) = x \cdot y(x)$, $y(1) = 1$ with the step size $h = 0.5$.

(b) Apply Φ_2 for the same initial value problem. Compute one step with $h = 0.5$.

(c) Determine the local truncation error and the order of the methode Φ_1.

(d) Show that applying Φ_2 for the model problem $y'(x) = \lambda x$ with $\lambda \in \mathbb{R}, \lambda < 0$ and $h > 0$ results in $|Y_{k+1}| \leq |Y_k|$.

Problem 34:
Compute numerical approximations for the initial value problem $y'(x) = -\sin(x)y(x)$ for $x \in [0, 20]$ with $y(0) = \exp(1)$. Use the embedded Runge-Kutta method

$$\begin{array}{c|ccc} 0 & & & \\ 1 & 1 & & \\ 1/2 & 1/4 & 1/4 & \\ \hline \text{order 2} & 1/2 & 1/2 & \\ \text{order 3} & 1/6 & 1/6 & 2/3 \end{array}$$

with stepsize control and the control parameter $\varepsilon = 10^{-3}$. Start with $h = 0.2$ for the first step.
Plot the applied stepsize in one plot and the error $|\exp(\cos(x_i)) - Y_i|$ in another one.

The tasks are intended both for processing in the seminars and for independent practice. Especially the 90 minutes of an exercise are sometimes not sufficient to discuss and work on all tasks.