Analysis and Numerics of Partial Differential Equations.

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Structure of the module

Lectures:

- Mon 9.00-11.00, room 17
- Thu 9.00-11.00, room 17 (until 30.04.) resp. Ex 04 (from 07.05.)
- slides under https://www.numerik.mathematik.uni-rostock.de/sawall/

Tutorials:

- by Tomass Andersons
- Thu 13.00 - 15.00, room 17

Contact:

- mathias.sawall@uni-rostock.de
- room 431, Ulmenstraße 69, Haus 3

Structure of the module

Exam:

- final examination of 120 min
- allowed are 10 leaves DIN-A4, hand written on both sides and a pocket calculator

Exercises:

- available under
  https://www.numerik.mathematik.uni-rostock.de/sawall/
- e. g. print them and think about them at home for your own
- the tasks are discussed in the tutorials

Web:

- https://www.numerik.mathematik.uni-rostock.de/sawall/
- Tutorials, slides, and more

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Outline

1. Introduction

2. Solution of first order partial differential equations

3. Solution of second order partial differential equation

4. High performance methods

Outline

What is a PDE?

Some applications:

1. one dimensional diffusion of a dye in water, $u(x,t)$ denotes the density, Fick's law for mass transport by diffusion results in the one dimensional diffusion equation or also called heat equation

$$\frac{\partial}{\partial t}u(x,t) = \alpha \frac{\partial^2}{\partial x^2}u(x,t)$$

2. density of cars $n(x,t)$ on a road with heavy traffic moving in one direction, with v being the speed of the cars results in the transport equation

$$\frac{\partial}{\partial t}n(x,t) + v\frac{\partial}{\partial x}n(x,t) = 0$$

What is a PDE?

Some applications:

3. a chain consisting of n elements, each with mass m connected by springs, for all single masses we get the discrete equations

$$m\ddot{u}_i = F_{i+1} - F_i = \beta(u_{i+1} - u_i - u_i + u_{i-1}), \quad i = 1, 2, \dots, n,$$

increasing n while decreasing β and m, results in the 1D wave equation

$$\frac{\partial^2}{\partial t^2} u(x, t) = c \frac{\partial^2}{\partial x^2} u(x, t),$$

4. the telegraph equation, where an infinitesimal piece of telegraph wire is modeled as an electrical circuit and has the form

$$u_{xx}(x, t) = LCu_{tt}(x, t) + (LG + RC)u_t(x, t) + RGu(x, t),$$

with the material properties resistance R, capacitance C and inductance L,

What is a PDE?

Some applications:

5. consider a crowd of n^2 accommodating people ordered in a square, each person has an opinion, given by the scalar p_{ij}, communication only with the immediate neighbours, assumption that everyone tries to minimize any conflict with the neighbours, willing to take an opinion that is the average of their opinions

$$p_{ij} = 0.25(p_{i-1,j} + p_{i+1,j} + p_{i,j-1} + p_{i,j+1})$$

information from the outside defines the boundary condition, for $n \to \infty$ this model goes over into the Laplace equation

$$p_{xx} + p_{yy} = 0$$

From Mattheij, Rienstra, ten Thije Boonkkamp, K.: Partial Differential Equations, 2005.

What is a PDE?

Several PDEs:

1. one-dimensional linear transport equation

$$u_t = -a u_x$$

   with the general solution $u(x,t) = u_0(x-at)$

2. one-dimensional nonlinear transport equation

$$u_t = f(u) u_x$$

3. Poisson-equation

$$\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = f(x_1, \ldots, x_n)$$

What is a PDE?

Several PDEs:

4. Laplace-equation

$$\Delta u = 0$$

5. Burger's equation

$$u_t + uu_x = \varepsilon u_{xx}$$

   and for $\varepsilon = 0$ the inviscid Burgers' equation

6. heat equation

$$u_t = k\Delta u$$

7. wave equation

$$u_{tt} = c^2 \Delta u$$

Notations

Partial derivative

$$\frac{\partial u}{\partial x} = u_x$$

Gradient for $u : \mathbb{R}^n \to \mathbb{R}$

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)^T$$

Divergence of the vector field $w : \mathbb{R}^n \to \mathbb{R}^n$

$$\nabla \cdot w = \sum_{i=1}^{n} \frac{\partial w_i}{\partial x_i}$$

Laplacian operator for $u : \mathbb{R}^n \to \mathbb{R}$

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

Boundary of a domain Ω is denoted by $\partial\Omega$

What is a PDE?

Definition 1.1

A partial differential equation (PDE) is an equation involving a function $u : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$ of several variables $(x_1, \dots, x_n)$ and its partial derivatives

$$F\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots\right) = 0.$$

Remarks:

- in general a PDE itself has no unique solution
- e. g. the solution of the transport equation $u_t = au_x$ is $u(x,t) = u_0(x - at)$ with $u_0 \in \mathcal{C}^1(\mathbb{R})$

Boundary conditions

Definition 1.2

1. *For* ***Dirichlet boundary conditions*** *the function is defined on the boundary as*

$$u(x) = g(x) \textit{ for } x \in \partial\Omega.$$

2. *In* ***Neumann-boundary conditions*** *holds*

$$\frac{\partial u}{\partial \overrightarrow{n}}(x) = 0 \textit{ for } x \in \partial\Omega$$

   with $\overrightarrow{n}$ *being the derivative in the direction of the outer unit normal at* $\partial\Omega$ *(short normal derivative).*

3. *For time-dependent PDEs next to boundary conditions we need* ***mixed initial boundary conditions*** *as*

$$u(x,0) = u_0(x) \textit{ for } x \in \Omega, \quad \textit{(for the heat equation),}$$
$$u_t(x,0) = v_0(x) \textit{ for } x \in \Omega, \quad \textit{(for the wave equation).}$$

Outline

1. Introduction
   - 1.1 What is a PDE?
   - 1.2 Classifications
   - 1.3 Finite differences and numerical differentiation
   - 1.4 Boundary value problem for second order ODE
   - 1.5 Discretization

Well-posedness

Cauchy-problems with PDEs can have:

- no solution or
- a unique solution or
- (infinite) many solution

Definition 1.3 (Hadamard's well-posedness condition)

A partial differential equation with boundary or initial and boundary conditions is called well-posed if

(a) *there exists a unique solution*

(b) *this solution depends continuosly on the data*

Otherwise the problem is called ill-posed.

Classification and charaterization

We classify PDEs by:

- the dimension n of the independent variable for

$$u : \mathbb{R}^n \to \mathbb{R}$$

- the dimension m of the dependent variable (scalar or system)

$$u : \mathbb{R}^n \to \mathbb{R}^m$$

- the order, this is the number of highest derivative
- linearity or not (semi-, quasi-linear, strong nonlinear)
- if the PDE is linear by constancy of coefficients
- its type (hyperbolic, parabolic, elliptic, mixed, changing)
- the applied boundary conditions (Dirichlet, Neumann, mixed, free)

Analytical and numerical methods

Analytical approaches:

- look for and provide general solutions and formulas
- make strong assumptions about the conditions and for the techniques
- work for PDEs and Cauchy-problems
- optionally specification of parameters and functions by initial conditions
- important issues are existence and uniqueness of the solution

Numerical methods:

- seek approximations to concrete solution
- work only for Cauchy-problems
- visualization and animation of the solution
- important issues are accuracy of the approach and stability of the scheme

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Finite differences

Definition 1.4

The approximations

$$D^+f(x) = \frac{f(x+h)-f(x)}{h}, \quad D^-f(x) = \frac{f(x)-f(x-h)}{h}, \quad D^0f(x) = \frac{f(x+h)-f(x-h}{2h}$$

for $f'(x)$ are called forward-, backward- and central difference.

Theorem 1.5 (Approximation orders)

For the finite differences holds

$$\begin{aligned} f'(x) - D^+f(x) &= \mathcal{O}(h), && \textit{(order 1)} \\ f'(x) - D^-f(x) &= \mathcal{O}(h), && \textit{(order 1)} \\ f'(x) - D^0f(x) &= \mathcal{O}(h^2). && \textit{(order 2)} \end{aligned}$$

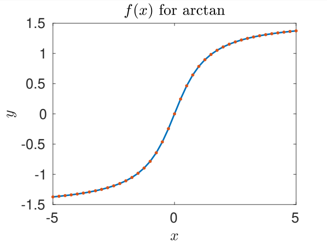
Thus, forward and backward differences are first order approximations to $f'(x)$ and the central difference is an approximation of second order.

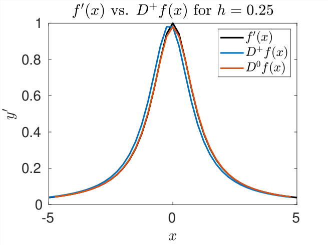
Errors

Example

Approximation of $f'(x_0)$ *for* $f(x) = \mathrm{arctan}(x)$ *and* $x_0 = 0.5$ *for different* h

h	$\lvert D^+f(x_0) - f'(x_0)\rvert$	$\lvert D^-f(x_0) - f'(x_0)\rvert$	$\lvert D^0f(x_0) - f'(x_0)\rvert$
0.1	$3.2 \cdot 10^{-2}$	$3.1 \cdot 10^{-2}$	$4.3 \cdot 10^{-4}$
0.01	$3.2 \cdot 10^{-3}$	$3.2 \cdot 10^{-3}$	$4.3 \cdot 10^{-6}$
10^{-5}	$3.2 \cdot 10^{-6}$	$3.2 \cdot 10^{-6}$	$4.8 \cdot 10^{-12}$.


$f(x)$ for arctan

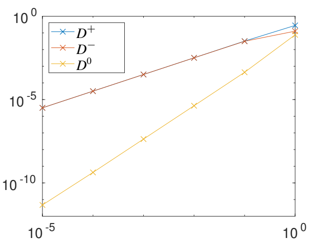

$f'(x)$ vs. $D^+f(x)$ for $h = 0.25$

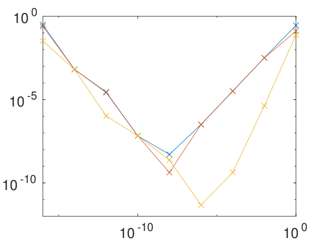
Errors

Example (Continuation)

But we cannot reduce h arbitrarily, as

h	$\lvert D^+f(x_0) - f'(x_0)\rvert$	$\lvert D^-f(x_0) - f'(x_0)\rvert$	$\lvert D^0f(x_0) - f'(x_0)\rvert$
10^{-12}	$2.7 \cdot 10^{-5}$	$2.9 \cdot 10^{-5}$	$1.0 \cdot 10^{-6}$
10^{-14}	$6.4 \cdot 10^{-4}$	$6.4 \cdot 10^{-4}$	$6.4 \cdot 10^{-4}$.





Problem: Catastrophic cancellation for $f(x_0 + h) - f(x_0 - h)$.

Finite difference for second order derivative

Definition 1.6

The approximation

$$D^{(2)}f(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

for $f''(x)$ is called central difference of second order.

The error of the central difference of second order is in $\mathcal{O}(h^2)$, so $D^{(2)}f$ is an approximation of second order.

Partial derivatives

Numerical differentiation in $\mathbb{R}^n$:

- let $u : \mathbb{R}^n \to \mathbb{R}$
- we need all in all n finite differences

$$\frac{u(x + (0, \dots, 0, h, 0, \dots, 0)^T) - u(x)}{h} \approx \frac{\partial u}{\partial x_i}(x), \qquad i = 1, \dots, n$$

Finite difference of second order:

- for $h_x = h_y = h$ holds

$$\begin{aligned} u_{xx}(x,y) &\approx \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2}, \\ u_{yy}(x,y) &\approx \frac{u(x,y-h) - 2u(x,y) + u(x,y+h)}{h^2}, \\ u_{xy}(x,y) &\approx \frac{u(x+h,y+h) - u(x+h,y-h) - u(x-h,y+h) + u(x-h,y-h)}{4h^2} \end{aligned}$$

- different stencils for $h_x \neq h_y$

5-point stencil

Definition 1.7

The 5-point stencil to approximate Δu *for* $u(x, y)$ *is*

$$\begin{aligned}\Delta u &= u_{xx} + u_{yy} \\ &\approx \frac{u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)}{h^2}.\end{aligned}$$

$$L_h^{(5)} = \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

Difference stencil for u_{xy}

$$\frac{1}{4h^2}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

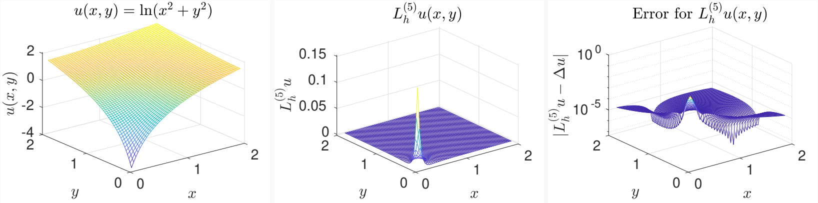
Applying the 5-point stencil

Example

Consider the harmonic function

$$u(x, y) = \ln(x^2 + y^2)$$

on $\Omega = [0.1, 2]^2$*. Using* $h = 0.01$ *we get (once again* $\Delta u = 0$*)*



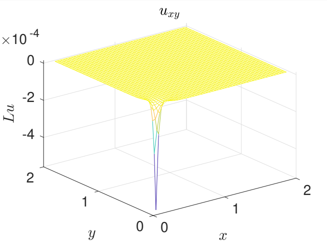
Errors for mixed partial derivative

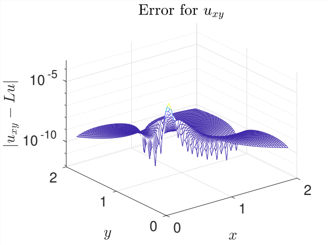
Example

Consider once agian

$$u(x,y) = \ln(x^2 + y^2)$$

on $\Omega = [0.1, 2]^2$*. The error for the approximation of* u_{xy} *and* $h = 0.01$ *are*





Outline

Boundary value problem of second order for ODE

Scalar linear boundary value problem of second order (with $\lambda \geq 0$):

$$\begin{aligned} -y''(x) + \lambda y(x) &= f(x_i), \quad a \leq x \leq b, \\ y(a) = \alpha, &\qquad y(b) = \beta \end{aligned}$$

Segmentation:

$$a = x_0 < x_1 < \ldots < x_N < x_{N+1} = b, \qquad x_j = a + jh, \qquad h = \frac{b-a}{m+1}$$

Boundary points:

$$(x_0, \alpha), \qquad (x_{N+1}, \beta)$$

Inner knots:

$x_1, \ldots, x_N$, to these knots we compute approximations $Y_i \approx y(x_i)$

Boundary value problem of second order for ODEs

Finite difference:

$$-y''(x_i) \approx \frac{-Y_{i-1} + 2Y_i - Y_{i+1}}{h^2}$$

Equations:

$$\begin{aligned}
i = 1: \quad & \frac{-\alpha + 2Y_1 - Y_2}{h^2} + \lambda Y_1 = f(x_1) \\
2 \leq i \leq N-1: \quad & \frac{-Y_{i-1} + 2Y_i - Y_{i+1}}{h^2} + \lambda Y_i = f(x_i) \\
i = N: \quad & \frac{-Y_{N-1} + 2Y_N - \beta}{h^2} + \lambda Y_N = f(x_N)
\end{aligned}$$

Boundary value problem of second order for ODE

System of linear equations:

$$\frac{1}{h^2}\begin{pmatrix} 2+\lambda h^2 & -1 & & & \\ -1 & 2+\lambda h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+\lambda h^2 & -1 \\ & & & -1 & 2+\lambda h^2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{pmatrix} = \begin{pmatrix} \alpha/h^2 + f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ \beta/h^2 + f(x_N) \end{pmatrix}$$

Outline

For a time dependent function

Grid for $u(x,t)$ with $x \in \Omega = [a,b]$ and $t \in [0, t_E]$:

$$x_i = a + ih,\ i = 0, \ldots, n, \quad h = \frac{b-a}{n}$$
$$t_j = jk,\ j = 0, \ldots, m, \quad k = \frac{t_E}{m}$$

Approximations:

$$U_{i,j} \approx u(x_i, t_j),\ i = 0, \ldots, n,\ j = 0, \ldots, m$$

Without a time dependency

Grid for $u(x,y)$ with $\Omega \subset \mathbb{R}^2$:

$$x_i = x_a + ih_x,\ i = 0,\ldots,n, \quad h_x = \frac{x_b - x_a}{n}$$
$$y_i = y_a + jh_y,\ j = 0,\ldots,n, \quad h_y = \frac{y_b - y_a}{m}$$

Approximations

$$U_{i,j} \approx u(x_i, y_j),\ i = 0,\ldots,n,\ j = 0,\ldots,m$$

Strict mathematically notation:

$$\Omega_h = \Omega \cap \left(h\mathbb{Z} \times h\mathbb{Z}\right), \qquad \Gamma_h = \partial\Omega_h, \qquad \bar{\Omega}_h = \Omega_h \cup \Gamma_h$$

For Dirichlet boundary conditions holds

$$U(x) = u(x), \quad \text{for } (x) \in \Gamma_h$$
$$U(x) \approx u(x), \quad \text{for } (x) \in \Omega_h$$

Discretized operator

Definition 1.8

For a second order PDE

$$Lu(x) = q(x), \quad \text{for } x \in \Omega$$

we call D_h the discretized operator if D_hU is a discretization for Lu.

Remark 1

1. *With D_h being the discretized operator, e. g. $L_h^{(5)}$ for Δu, holds for the PDE $Lu = q$ that*

$$D_hU(x) = q(x) \quad \text{for } (x) \in \Omega_h.$$

2. *In general, the exact solution does not satisfy the discretized PDE, thus*

$$LU(x) = q(x) + e(x, h).$$

Outline

1. Introduction
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Quasilinear first order partial differential equations

Quasilinear first order PDE:

- for $z : \mathbb{R}^n \to \mathbb{R}$ it has the form

$$\sum_{k=1}^{n} P_k(x, z) \frac{\partial z}{\partial x_k} = R(x, z)$$

  with $P_k : \mathbb{R}^{n+1} \to \mathbb{R}$ for $k = 1, \ldots, n$ as well as $R : \mathbb{R}^{n+1} \to \mathbb{R}$
- for $n = 2$ we use x, y instead of x_1, x_2 and $P(x, y, z)$, $Q(x, y, z)$
- then the PDE has the form

$$P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z)$$

- the coefficients may depend lso on the unknown function $z(x, y)$ next to x and y

Outline

Homogeneous linear PDE in two variables

Consider the homogeneous linear PDE (most simple case)

$$P(x,y)z_x + Q(x,y)z_y = 0. \tag{1}$$

Theorem 2.1

Let $u(x,y)$ *be the solution of*

$$\frac{\mathrm{d}x}{P(x,y)} = \frac{\mathrm{d}y}{Q(x,y)}. \tag{2}$$

If we write the solution as $u(x,y) = C$ *(this is the so called characteristic curve) with an arbitrary constant* $C \in \mathbb{R}$*, then the set of solutions of (1) consists of all functions*

$$z = f(u(x,y))$$

with an arbitrary differentiable function $f : \mathbb{R} \to \mathbb{R}$*.*

Homogeneous linear PDE in two variables

Example

For the PDE

$$az_x + bz_y = 0$$

with constants $a, b \in \mathbb{R}$ *we get*

$$\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} \quad \Leftrightarrow \quad \int b \,\mathrm{d}x = \int a \,\mathrm{d}y \quad \Leftrightarrow \quad bx + C = ay$$

and thus $ay - bx = C$. *The solution of the PDE reads*

$$z(x, y) = f(ay - bx)$$

with an arbitrary function $f : \mathbb{R} \to \mathbb{R}$. *The function* f *is determined by an optional initial condition.*

Homogeneous linear PDE in two variables

Example

Consider the initial value problem

$$\sqrt{1-x^2}z_x + z_y = 0, \qquad z(0,y) = y.$$

We get

$$\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{\mathrm{d}y}{1} \quad \Leftrightarrow \quad \arcsin(x) = y + C$$

and it holds

$$\begin{aligned} u(x,y) &= C = \arcsin(x) - y \\ z(x,y) &= f(u(x,y)) = f(\arcsin(x) - y). \end{aligned}$$

The initial condition results in

$$\begin{aligned} f(\arcsin(0) - y) &= y \\ f(-y) &= y \\ z(x,y) &= y - \arcsin(x). \end{aligned}$$

Homogeneous linear PDE in two variables

Remark 2 (Isolines)

Let the function $f : \Omega \to \mathbb{R}$ *with* $\Omega \subset \mathbb{R}^n$ *be differentiable. In all points* $x \in \Omega$ *of the domain, where* $\nabla f(x)$ *does not vanish, the set*

$$\{y \in \Omega : \; f(y) = f(x)\}$$

is locally a curve orthogonal to $\nabla f(x)$*.*

Geometric considerations:

- let the coefficients $P(x,y)$ and $Q(x,y)$ be the components of the vector field
$$\overrightarrow{v}(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$
- for the PDE holds $P(x,y)z_x + Q(x,y)z_y = (\overrightarrow{v}, \nabla z)_2 = 0$
- so the gradient of z is orthogonal to the vector $\overrightarrow{v}(x,y)$
- next to $\overrightarrow{v}$ also the isolines to a given constant of $z(x,y)$ are orthogonal to $\nabla z(x,y)$
- the curves $z(x,y) = C$ are tangent to the vector field $\overrightarrow{v}(x,y)$ for all (x,y)

Geometric considerations

Example

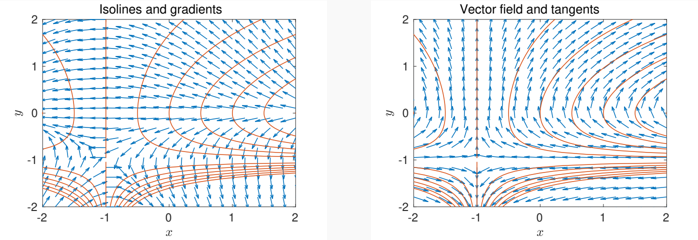
The PDE $(x+1)yz_x + (y+1)z_y = 0$ *has the vector field*

$$\vec{v} = \begin{pmatrix} (x+1)y \\ y+1 \end{pmatrix}.$$

The tangents are

$$\ln(x+1) + \ln(y+1) - y = C$$

and the solution reads $z(x,y) = f(\ln(x+1) + \ln(y+1) - y)$ *with an arbitrary* $f \in \mathcal{C}(\mathbb{R})$*.*

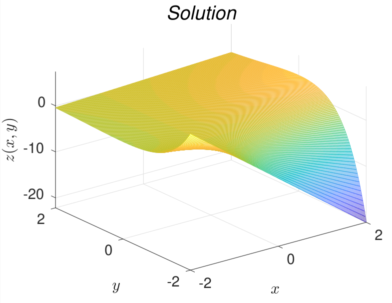


Geometric considerations

Example (Continuation)

If we claim the initial condition $z(x,0) = x+1$*, then we get the unique solution*

$$z(x,y) = (x+1)(y+1)\mathbf{e}^{-y}.$$



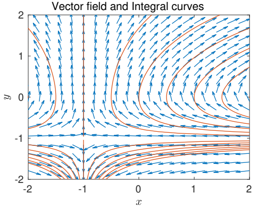
Integral curves

Definition 2.2

Let $\vec{v} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a vector field with* $\vec{v}(x,y) = (P(x,y), Q(x,y))^T$. *A curve* C *which is the solution of*

$$\frac{\mathrm{d}x}{P(x,y)} = \frac{\mathrm{d}y}{Q(x,y)}$$

is called an integral curve or characteristic curve of the PDE (1) if C *is tangent to* $\vec{v}$ *at every point of* C.



Outline

Integral curves and surfaces

Integral curves and surfaces for 3-dimensional vector fields:

- consider now the 3-dimensional vector field

$$\vec{v}(x,y,z) = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$$

- integral curves of $\vec{v}$ are the key to solve inhomogeneous quasi-linear in 2D as well as homogeneous quasi-linear first order PDEs in 3D
- in 3D, there are two independent orthogonal directions.

Definition 2.3

A curve or a surface is called an integral curve or an integral surface to a given vector field $\vec{v}$, if it is a tangent to it at every point.

Integral curves and surfaces

Remark 3

If we find two independent integral surfaces to a given $\vec{v}$*, their intersection will be an integral curve of* $\vec{v}$*. If we succeed to define integral surfaces in implicit form* $u_1(x, y, z) = C_1$ *and* $u_2(x, y, z) = C_2$*, the general solution to the PDE*

$$\vec{v} \cdot \nabla u = 0$$

can be represented in the form

$$F\big(u_1(x, y, z), u_2(x, y, z)\big) = 0$$

with a differentiable function F*.*

Procedures to solve linear first order PDEs

Solution of the linear inhomogeneous PDE in two variables:

1. computation of two functionally independent integral curves $C_1 = u_1(x, y, z)$ and $C_2 = u_2(x, y, z)$ for the vector field,
2. representation of one integral curve by the other as $u_2(x, y, z) = f(u_1(x, y, z))$ and isolation of z.

Solution of the linear homomgenous PDE in three variables:

1. computation of two functionally independent integral curves $C_1 = u_1(x, y, z)$ and $C_2 = u_2(x, y, z)$ for the vector field,
2. the solution u has the form $u(x, y, z) = f(u_1(x, y, z), u_2(x, y, z)$.

Solution of the linear inhomogeneous PDE in two variables

Theorem 2.4

The function $z = z(x, y)$ *is a solution of the PDE*

$$P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z) \tag{3}$$

if and only if the surface $z = z(x, y)$ *is an integral surface of the vector field* $\overrightarrow{v} = (P, Q, R)$*.*

Solution of the linear inhomogeneous PDE in two variables

Example

Consider the initial value problem

$$2xyz_x + (y^2+1)z_y = 4xy^3, \qquad z(x,0) = x+1.$$

The vector field is

$$\overrightarrow{v}(x,y,z) = \begin{pmatrix} 2xy \\ y^2+1 \\ 4xy^3 \end{pmatrix}$$

and results in the equations

$$\frac{\mathrm{d}x}{2xy} = \frac{\mathrm{d}y}{y^2+1} = \frac{\mathrm{d}z}{4xy^3}.$$

Solution of the linear inhomogeneous PDE in two variables

Example (Continuation)

We take the first equation (find the integral curves of the 2D vector field by separating the variables x *and* y*) and get*

$$\begin{aligned}\frac{\mathrm{d}x}{x} = \frac{2y}{y^2+1}\,\mathrm{d}y \quad &\Leftrightarrow \quad \ln(x) = \ln|y^2+1| + C\\ &\Leftrightarrow \quad x = \underbrace{\exp(C)}_{\tilde{C}}\,|y^2+1|\\ &\Leftrightarrow \quad \tilde{C} = \frac{x}{y^2+1}.\end{aligned}$$

For the second integral surface (geometrically we integrate z *along the characteristics) we use*

$$\begin{aligned}\frac{\mathrm{d}y}{y^2+1} = \frac{\mathrm{d}z}{4xy^3} \quad &\Leftrightarrow \quad \mathrm{d}z = \frac{4xy^3}{y^2+1}\,\mathrm{d}y \quad \Leftrightarrow \quad \mathrm{d}z = 4\tilde{C}y^3\,\mathrm{d}y\\ &\Leftrightarrow \quad z = \tilde{C}y^4 + C \quad \Leftrightarrow \quad z = \frac{xy^4}{y^2+1} + C.\end{aligned}$$

Solution of the linear inhomogeneous PDE in two variables

Example (Continuation)

The integral curves are

$$u_1(x,y,z) = \frac{x}{y^2+1}, \qquad u_2(x,y,z) = z - \frac{xy^4}{y^2+1}.$$

Finally, we have to combine both results and represent u_2 *by* u_1 *and use the initial condition to determine* f

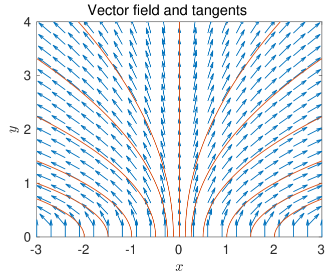
$$\begin{aligned} u_2(x,y,z) = f(u_1(x,y,z)) \quad &\Leftrightarrow \quad z - \frac{xy^4}{y^2+1} = f\Big(\frac{x}{y^2+1}\Big) \\ &\Leftrightarrow \quad z = \frac{xy^4}{y^2+1} + f\Big(\frac{x}{y^2+1}\Big) \\ x+1 = z(x,0) \quad &\Leftrightarrow \quad x+1 = f(x) \quad \Rightarrow \quad f(v) = v+1 \end{aligned}$$

and the final solution reads

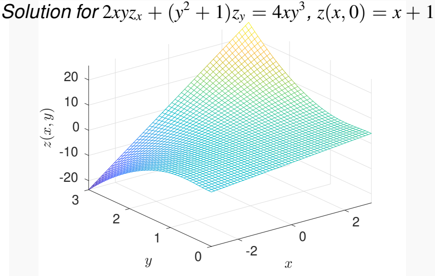
$$z = \frac{x}{y^2+1} + 1 + \frac{xy^4}{y^2+1}.$$

Solution of the linear inhomogeneous PDE in two variables

Example (Continuation)



Vector field and tangents



Solution for $2xyz_x + (y^2+1)z_y = 4xy^3$, $z(x,0) = x+1$

Solution of the linear homomgenous PDE in three variables

Theorem 2.5

The function $u(x,y,z)$ *is a solution of the PDE*

$$P(x,y,z)u_x + Q(x,y,z)u_y + R(x,y,z)u_z = 0 \tag{4}$$

if and only if the surface $u(x,y,z) = c$ *is an integral surface of the vector field* $\overrightarrow{v} = (P,Q,R)^T$*.*

Relation between 2D and 3D:

- solving both PDEs from (3) and (4) using the vector field $\overrightarrow{v}(x,y,z) = (P,Q,R)^T$
- integral surfaces in the explicit form $z = z(x,y)$ for (3)
- general form $u(x,y,z) = C$ for (4)

Outline

Integral curves in parametric form

Theorem 2.6

If C is an integral curve for the vector field $\overrightarrow{v}$, then there exists a parametric form

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad \textit{for } t \in I \subset \mathbb{R}$$

with $x(t)$, $y(t)$ and $z(t)$ being the solutions of the ODEs

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = P(x,y,z), \quad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y,z), \quad \dot{z} = \frac{\mathrm{d}z}{\mathrm{d}t} = R(x,y,z).$$

Procedure:

- solve all three ODEs to compute the parametric form of vector-field
- infinitely many parametric forms of the integral curves exist.

Integral curves in parametric form

Example

Consider the vector field $\overrightarrow{v} = (y, -x, 0)^T$*. The associated ODEs are*

$$\dot{x} = y, \quad \dot{y} = -x, \quad \dot{z} = 0.$$

We combine the first two equations and get

$$\begin{aligned} &\ddot{x} = \dot{y} = -x \quad \Rightarrow \quad x = C_1 \sin(t) + C_2 \cos(t), \\ &y = \dot{x} = C_1 \cos(t) - C_2 \sin(t). \end{aligned}$$

Further, the equation $\dot{z} = 0$ *results in* $z = C_3$ *and the parametric form is*

$$\begin{aligned} x(t) &= C_1 \sin(t) + C_2 \cos(t), \\ y(t) &= C_1 \cos(t) - C_2 \sin(t), \\ z(t) &= C_3. \end{aligned}$$

As these are circles in the z*-plane the integral curves in parametric form are*

$$x^2 + y^2 = \tilde{C}_1, \quad z = C_3.$$

Integral curves in Cartesian coordinates

Solve two of the equations

$$\frac{\mathrm{d}x}{\mathrm{d}P(x,y,z)} = \frac{\mathrm{d}y}{\mathrm{d}Q(x,y,z)} = \frac{\mathrm{d}z}{\mathrm{d}R(x,y,z)} = \mathrm{d}t$$

Special case:

- $P = P(x,y)$ and $Q = Q(x,y)$ are independent of z
- then the PDE is a semilinear and we start with

$$\frac{\mathrm{d}x}{P(x,y)} = \frac{\mathrm{d}y}{Q(x,y)}$$

- under some circumstances we can get representations

$$u_1(x,y) = C_1, \quad u_2(x,y,z) = C_2$$

of the integral curves with arbitrary constants $C_1, C_2 \in \mathbb{R}$.

Integral curves in Cartesian coordinates

Example

Consider the vector field $\overrightarrow{v} = (x, y, xy(z^2+1)/z)^T$*. The equations are*

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{xy(z^2+1)/z}$$

and the first one results in

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}x}{y} \quad \Rightarrow \quad \ln|x| + C = \ln|y| \quad \Rightarrow \quad y = C_1 x, \text{ resp. } \frac{y}{x} = C_1.$$

Using this we get for the next one

$$\begin{aligned} \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}z}{xy(z^2+1)/z} = \frac{\mathrm{d}z}{C_1 x^2(z^2+1)/z} \quad &\Leftrightarrow \quad 2C_1 x\,\mathrm{d}x = \frac{2z}{z^2+1}\,\mathrm{d}z \\ &\Leftrightarrow \quad C_1 x^2 = \ln|z^2+1| + C_2 \end{aligned}$$

and the integral curves are

$$u_1(x,y,z) = \frac{y}{x}, \quad u_2(x,y,z) = xy - \ln|z^2+1|.$$

Outline

Solution of homogeneous linear PDE in three dimensions

Consider the PDE

$$P(x,y,z)u_x + Q(x,y,z)u_y + R(x,y,z)u_z = 0. \tag{5}$$

Theorem 2.7

1. *Let*

   $$u_1(x,y,z) = C_1, \quad u_2(x,y,z) = C_2$$

   define the set of integral curves of the vector field $\overrightarrow{v}(x,y,z) = (P,Q,R)^T$. Then $u_1(x,y,z)$ and $u_2(x,y,z)$ solve the PDE from (5).
2. *Let $u_1(x,y,z)$ and $u_2(x,y,z)$ be two arbitrary solutions of (5) and let $f(u_1,u_2)$ be an arbitrary differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$. Then*

   $$u(x,y,z) = f\big(u_1(x,y,z), u_2(x,y,z)\big)$$

   is also a solution of (5).

Solution of homogeneous linear PDE in three dimensions

Theorem 2.8

If $u_1(x,y,z)$ *and* $u_2(x,y,z)$ *solve* $P(x,y,z)u_x + Q(x,y,z)u_y + R(x,y,z)u_z = 0$ *and*

$$(\nabla u_1 \times \nabla u_2)(x_0, y_0, z_0) \neq 0,$$

then

1. $u_1(x,y,z) = C_1$ *and* $u_2(x,y,z) = C_2$ *define the set of all integral curves of the vector field* $\overrightarrow{v}$ *at* (x_0, y_0, z_0).
2. *for any other solution* $u(x,y,z)$ *there exists a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *such that*

   $$u(x,y,z) = f(u_1(x,y,z), u_2(x,y,z))$$

   in a neighbourhood U *around* (x_0, y_0, z_0).

Solution of homogeneous linear PDE in three dimensions

Procedure to solve a PDE as in (5):

1. compute the integral curves by solving either

   $$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x,y,z), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y,z), \quad \frac{\mathrm{d}z}{\mathrm{d}t} = R(x,y,z)$$

   for the parametric form or

   $$\frac{\mathrm{d}x}{P(x,y,z)} = \frac{\mathrm{d}y}{Q(x,y,z)} = \frac{\mathrm{d}z}{R(x,y,z)}$$

   for the Cartesian representation
2. the general solution of the PDE reads

   $$u(x,y,z) = f\big(u_1(x,y,z), u_2(x,y,z)\big)$$

   with an arbitrary differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$
3. (optionally) determine the function f by the initial data

Solution of homogeneous linear PDE in three dimensions

Example

Consider the PDE

$$2xyu_x + (y^2 + 1)u_y + 4xy^3u_z = 0$$

with the vector field

$$\overrightarrow{v}(x, y, z) = \big(2xy,\ y^2 + 1,\ 4xy^3\big)^T$$

as some examples ago. The integral curves are

$$u_1(x, y, z) = \frac{x}{y^2 + 1}, \qquad u_2(x, y, z) = z - \frac{xy^4}{y^2 + 1}$$

and the solution is

$$u(x, y, z) = f\Big(\frac{x}{y^2 + 1}, z - \frac{xy^4}{y^2 + 1}\Big).$$

Optionally: check whether $u(x, y, z)$ *fulfills the PDE with*

$$u_x = -\frac{1}{y^2 + 1}f_{u_1} - \frac{y^4}{y^2 + 1}f_{u_2}, \ldots.$$

Outline

2. Solution of first order partial differential equations

Initial value problems for first order PDEs in 2D

Quasilinear PDE in two-dimensional space

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z) \tag{6}$$

The solution should contain a curve C defined as

$$x(t) = \phi(t), \quad y(t) = \psi(t), \quad z(t) = \chi(t)$$

Thus it should hold

$$z\big(\phi(t), \psi(t)\big) = \chi(t). \tag{7}$$

Initial value problems for first order PDEs in 2D

Example

Consider the initial value problem

$$xz_x + yz_y = z, \qquad z(t+1,t) = t^2.$$

We have to solve

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{z}$$

and integration results in

$$u_1 = \frac{y}{x}, \quad u_2 = \frac{z}{x}.$$

We can represent z *by* u_2 *to get the general solution as*

$$u_2(x,y,z) = f(u_1(x,y,z)) \quad \Rightarrow \quad z = xf(\frac{y}{x}).$$

Initial value problems for first order PDEs in 2D

Example (Continuation)

To compute f *such that* $z(x,y)$ *contains the curve* C *defined by* $(\phi(t),\psi(t),\chi(t))$ *we use*

$$z(t+1,t)=t^2 \quad \Rightarrow \quad t^2=(t+1)f(\frac{t}{t+1}).$$

The coordinate transformation $v=t/(t+1)$ *and thereby* $t=v/(1-v)$ *results in*

$$f(v)=\frac{v^2}{1-v}.$$

Thus the solution is

$$z(x,y)=xf(\frac{y}{x})=\frac{x\left(\frac{y}{x}\right)^2}{1-\frac{y}{x}}=\frac{y^2}{x}\cdot\left(\frac{1}{1-\frac{y}{x}}\right)=\frac{y^2}{x-y}.$$

The next Theorem is about to decide in general whether there exists a solution or not and whether a potential solution is unique or not.

Initial value problems for first order PDEs in 2D

Theorem 2.9

Consider the initial value problem

$$P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z), \qquad z\big(\phi(t),\psi(t)\big) = \chi(t) \quad \text{for } t \in I.$$

with $\phi, \psi, \chi \in \mathcal{C}(\mathbb{R})$ *and* $(x_0, y_0, z_0) = (\phi(t_0), \psi(t_0)\chi(t_0))$ *being a point on the curve* C*.*

1. *The IVP has a uniquely determined solution* $z(x,y)$ *in a neighbourhood of* (x_0, y_0) *if*

$$\frac{\phi'(t_0)}{P(x_0,y_0,z_0)} \neq \frac{\psi'(t_0)}{Q(x_0,y_0,z_0)}.$$

2. *The IVP has no solution in a neighbourhood of* (x_0, y_0) *if*

$$\frac{\phi'(t_0)}{P(x_0,y_0,z_0)} = \frac{\psi'(t_0)}{Q(x_0,y_0,z_0)} \neq \frac{\chi'(t_0)}{R(x_0,y_0,z_0)}.$$

3. *The IVP has infinitely many solutions in a neighbourhood of* (x_0, y_0) *if*

$$\frac{\phi'(t_0)}{P(x_0,y_0,z_0)} = \frac{\psi'(t_0)}{Q(x_0,y_0,z_0)} = \frac{\chi'(t_0)}{R(x_0,y_0,z_0)}.$$

Initial value problems for first order PDEs in 2D

Example

Consider the PDE $xz_x + yz_y = z$ *and the initial condition*

(a) $z(t+1, t) = t^2$,

(b) $z(t, t) = t^2$ *resp.*

(c) $z(t, t) = t$.

For (a) we get

$$\phi'(t) = 1, \qquad \psi'(t) = 1, \qquad \chi'(t) = 2t$$

as well as

$$\frac{\phi'(t)}{P(x,y,z)} = \frac{1}{t+1}, \qquad \frac{\psi'(t)}{Q(x,y,z)} = \frac{1}{t}$$

and thereby this initial condition results in a unique solution, see slides 66 and 67.

Initial value problems for first order PDEs in 2D

Example

Consider the PDE $xz_x + yz_y = z$ *and the initial condition*

(a) $z(t+1,t) = t^2$,

(b) $z(t,t) = t^2$ *resp.*

(c) $z(t,t) = t$.

For (b) the fractions are

$$\frac{\phi'(t)}{P(x,y,z)} = \frac{1}{t}, \quad \frac{\psi'(t)}{Q(x,y,z)} = \frac{1}{t}, \quad \frac{\chi'(t)}{R(x,y,z)} = \frac{2t}{t^2} = \frac{2}{t}$$

and there exists no solution.

Initial value problems for first order PDEs in 2D

Example

Consider the PDE $xz_x + yz_y = z$ *and the initial condition*

(a) $z(t+1, t) = t^2$,

(b) $z(t, t) = t^2$ *resp.*

(c) $z(t, t) = t$.

For (c) the fractions are

$$\frac{\phi'(t)}{P(x,y,z)} = \frac{\psi'(t)}{Q(x,y,z)} = \frac{\chi'(t)}{R(x,y,z)} = \frac{1}{t}.$$

Thus infinitely many solutions exist.

Inviscid Burgers' equation

Example

Consider inviscid Burgers' equation

$$u_t + uu_x = 0, \qquad u(x,0) = v(x)$$

with $x \in \mathbb{R}$ *and* $t > 0$*. The integral curves are*

$$C_1 = u, \qquad C_2 = x - C_1 t = x - ut.$$

Replace C_2 *by* C_1 *to determine* $f(v)$

$$\begin{aligned} C_2 = f(C_1) \quad &\Leftrightarrow \quad u(x,t) = f(x - u(x,t)t) \\ u(x,0) = v(x) \quad &\Rightarrow \quad f(x) = v(x) \end{aligned}$$

and the solution reads

$$u(x,t) = v(x - u(x,t)t)$$

which is an implicit definition of u*.*

Outline

2. Solution of first order partial differential equations

Upwind scheme

Consider the PDE

$$u_t = -vu_x$$

Forward difference in time and backward difference in space direction with $U_{i,j} \approx u(x_i, t_j)$ result in

$$\frac{U_{i,j+1} - U_{i,j}}{k} = -v\frac{U_{i-1,j} - U_{i,j}}{h}.$$

Initialization

$$U_{i,0} = u_0(x_i), \quad i = 1, \dots, n,$$

Iteration

$$U_{i,j+1} = U_{i,j} - v\frac{k}{h}\big(U_{i-1,j} - U_{i,j}\big). \tag{8}$$

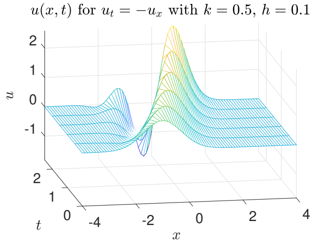
Upwind scheme

Example

For

$$u_t = -vu_x, \qquad u(x,0) = u_0(x) = \exp(-(x+1)^2)$$

the scheme from above using with $v = 1$, $h = 0.1$ *and* $k = 0.5$ *on* $x \in [-5,5]$ *and* $t \in [0,2.5]$ *delivers*

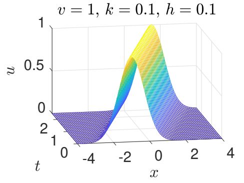


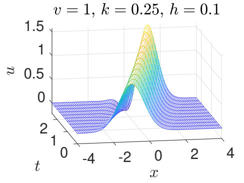
The scheme does not work at all.

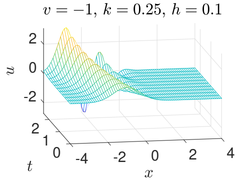
Upwind scheme

Example

Changing the time step size to $k \in \{0.1,\ 0.25\}$ *deliver the first two results, the last one is for* $v = -11$*.*







Only the results from the left plot is acceptable.

Analysis

What is the problem?

- analytical solution of the Cauchy problem is

$$u(x,t) = u_0(x - vt), \quad u(x,t) = \exp(-(x - t + 1)^2)$$

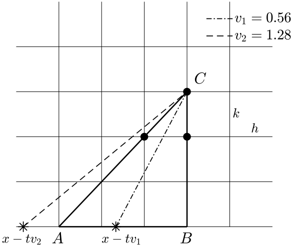
- the solution at a certain point (x,t) dependents on the initial data at $(x - vt, 0)$
- numerical domain of dependence for the simple upwind scheme

$$x \in [x - t\frac{h}{k}, x]$$

- numerical domain of dependence must include the dependence from initial data

$$x - vt \in \left[x - t\frac{h}{k}, x\right].$$

Stability condition - domain of dependence



Theorem 2.10 (Courant-Friedrich-Levy (CFL) condition)

For $u_t = vu_x$ with $v > 0$ the numerical solution using the simple upwind scheme from (8) is stable if and only if the *stability condition*

$$c = \frac{vk}{h} \leq 1.$$

holds.

Accuracy

Computation of the accuracy for the PDE

$$u_t - v u_x = f.$$

Finite differences result in

$$\frac{u(x,t+k) - u(x,t)}{k} + \mathcal{O}(k) - v\frac{u(x,t) - u(x-h,t)}{h} + \mathcal{O}(h) = f.$$

The error is in $\mathcal{O}(k+h)$, the upwind scheme is first order consistent in t and x direction.

Outline

2. Solution of first order partial differential equations
   - 2.1 Homogeneous linear PDE in two variables
   - 2.2 Integral curves and surfaces for 3-dimensional vector fields
   - 2.3 Calculation of integral curves
   - 2.4 Solution of homogeneous linear PDE in three dimensions
   - 2.5 Initial value problems for first order PDEs in 2D
   - 2.6 Upwind scheme
   - 2.7 Lax-Friedrich scheme
   - 2.8 Lax-Wendroff scheme
   - 2.9 Convergence of the methods

Lax-Friedrich scheme

Definition 2.11

Consider the PDE $u_t + vu_x = 0$*. The* *Lax-Friedrich-scheme* *reads*

$$U_{i,j+1} = \frac{U_{i-1,j} + U_{i+1,j}}{2} - v\frac{k}{2h}\big(U_{i+1,j} - U_{i-1,j}\big).$$

Reasonable results for negative as well as for positive speeds v, unless the step size ratio becomes too large.

Remark 4

1. *The Lax-Friedrich scheme is stable if the CFL-condition*

   $$\left|v\frac{k}{h}\right| \leq 1$$

   is fulfilled. Thus the condition gives an upper bound on k *for known* v *and* h*.*
2. *The Lax-Friedrich scheme is of order 1 in time direction and order 2 in* x*-direction.*

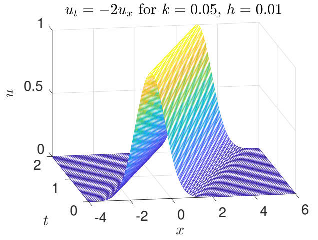
Lax-Friedrich scheme

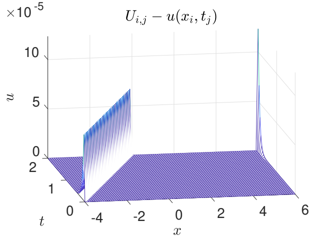
Example

We apply the scheme to

$$u_t = -2u_x,$$
$$u(x,0) = u_0(x) = \exp(-(x+1)^2).$$

The results are



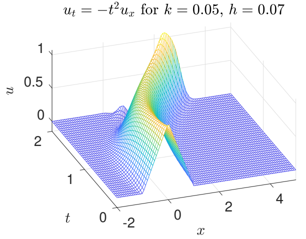


Lax-Friedrich scheme

Example

Consider the transport equation with variable speed $u_t - t^2 u_x = 0$ *and* $u_0(x) = u(x,0) = \max(0, 1 - |x|)$.

We use $k = 0.05$ *and* $h = 0.07$ *and compute approximations for* $t \in [0,2]$.



Slight instability at the end, as the step size ration is $k/h = \frac{5}{7}$ *but for* $t > \sqrt{7/5}$ *holds*

$$\left| v \frac{k}{h} \right| = t^2 \frac{5}{7} > 1$$

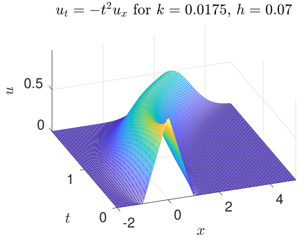
Lax-Friedrich scheme

Example

For $t \in [0,2]$ the maximal time step size is

$$h_{\max} = \frac{h}{t_{end}^2} = 0.0175.$$

Therefore the results are



Lax-Friedrich

Implementation of Lax-Friedrich using matrices:

```
u0 = @(x) max(0,1-abs(x));

h = 0.05;
X = -2:h:5;

k = 0.05;
T = 0:k:2;

v = 1;
U(:,1) = u0(X');

for i=1:length(T)-1
   du = [0; U(3:end,i)-U(1:end-2,i); 0]/2/h;
   U(:,i+1) = [0; (U(3:end,i)+U(1:end-2,i))/2; 0]-v*k*du;
end
mesh(X,T,U')
```

Outline

Lax-Wendroff scheme

Lax-Wendroff scheme:

- scheme of order 2 in time direction
- Taylor expansion

$$u(x, t+k) = u(x,t) + k u_t(x,t) + \frac{k^2}{2} u_{tt}(x,t) + \mathcal{O}(k^3)$$

- using $u_t = -v u_x \; u_{tx} = -(v u_x)_x$ results in

$$u_{tt} = \frac{\partial}{\partial t}(-v u_x) = -v u_{tx} = v^2 u_{xx}$$

Definition 2.12

For constant v the Lax-Wendroff scheme is

$$U_{i,j+1} = U_{i,j} - v\frac{k}{2h}\left(U_{i+1,j} - U_{i-1,j}\right) + \frac{v^2 k^2}{2h^2}\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right).$$

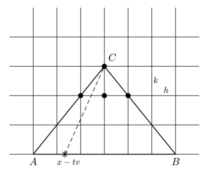
Lax-Wendroff scheme

Remark 5

1. *Due to the construction the accuracy is in time direction 2 as well as in* x*-direction. Thus the error is in* $\mathcal{O}(h^2 + k^2)$*.*
2. *The stability condition remain unchanged. The Lax-Wendroff stencil is stable if*

$$\left| v \frac{k}{h} \right| \leq 1.$$

The geometric basis is the same for Lax-Friedrich as well as for Lax-Wendroff and looks



Lax-Wendroff scheme

Implementation:

```
u0 = @(x) max(0,1-abs(x));

h = 0.05;
X = -2:h:5;

k = 0.05;
T = 0:k:2;

v = 1;
U(:,1) = u0(X');

for i=1:length(T)-1
   du = [0; U(3:end,i)-U(1:end-2,i); 0]/2/h;
   ddu = [0; U(1:end-2,i)-2*U(2:end-1,i)+U(3:end,i); 0]/h^2;
   U(:,i+1) = U(:,i)-v*k*du+v^2*k^2/2*ddu;
end
mesh(X,T,U')
```

Lax-Wendroff scheme

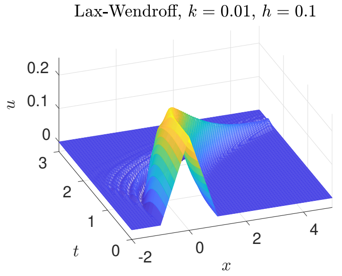
Example

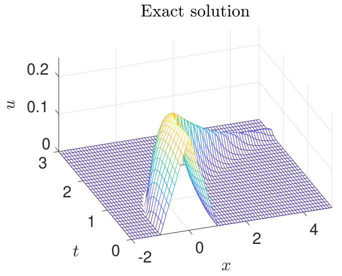
For $x \in [-2,5]$, $t \in [0,3]$ *consider the inhomogeneous PDE with variable coefficient*

$$u_t + t^2 u_x = -xu, \qquad u(x,0) = \max(0, 1-|x|)/4$$

The step sizes are $h = 0.1$ *and* $k = 0.01$. *The analytical solution is*

$$u(x,t) = \exp\left(0.25t^4 - tx\right) \frac{\max(0, 1 - |\frac{1}{3}t^3 - x|)}{4}.$$





Outline

Convergence of the methods

Convergence of numerical methods for ODEs

$$\begin{array}{l} \ \text{consistency} \\ +\ \text{stability} \\ \hline =\ \text{convergence} \end{array}$$

Ingredients:

- focus on the homogeneous transport equation $u_t + vu_x = 0$
- analysis is done for the natural norm

$$\|u\| = \int_{\mathbb{R}} |u(x)| \, \mathrm{d}x$$

- l_1-norm for the discretized solution on a grid

$$\|U(:,j)\| = h \sum_{i=1}^{n} |U_{i,j}|.$$

Consistency

Progress in time direction depending on the step sizes h and k

$$U(:,j+1) = HU(:,j)$$

Definition 2.13

The local truncation error of a scheme is

$$\tau(x,t,k) = \frac{1}{k}\Big(u(x,t+k) - HU(:,j)\Big).$$

A scheme is called consistent, if $\lim_{k\to 0} \|\tau(x,t,k)\| = 0$.

Definition 2.14

A method is called of order of consistency q, if for all smooth initial functions with compact support there is a constant C such that

$$\|\tau(x,t,k)\| \leq Ck^q$$

for sufficient small k and $t \leq T-k$.

Stability from step to step

Definition 2.15

A method is called <u>*stable*</u>*, if for all* T *there exists a constant* C_S *and an* $k_0 > 0$*, such that*

$$\|H^j\| \leq C_S \quad \textit{for all } jk \leq T,\ k < k_0.$$

Sufficient is

$$\|H\| \leq 1 + \alpha k$$

as this implies

$$\|H^j\| \leq (1 + \alpha k)^j \leq \exp(\alpha k j) \leq \exp(\alpha T).$$

Convergence

For the analysis of the convergence let $U^{(k)}$ be the numerical solution of the PDE for time step size k and given initial conditions, extended to a piecewise linear function for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ and let u be the exact solution. The error E_k is defined as

$$E_k(x,t) = U^{(k)}(x,t) - u(x,t).$$

Definition 2.16

A method is called <u>*convergent*</u>*, if for suitable norm and all proper* $u_0(x)$ *for each* $t > 0$ *holds*

$$\lim_{k \to 0} \|E_k(\cdot,t)\| = \lim_{k \to 0} \int_{\mathbb{R}} |E(x,t)| \, \mathrm{d}x = 0.$$

Convergence

Theorem 2.17 (Lax equivalence theorem)

For a consistent linear method stability is necessary and sufficient for convergence.

Proof: It holds

$$\begin{aligned}\|E_k(\cdot,t_j)\| &\le \|H\|\|E_k(\cdot,t_{j-1})\| + k\|\tau(\cdot,t_{j-1},k)\| \\ &\le \|H^j\|\|E_k(\cdot,0)\| + k\sum_{l=1}^{j}\|H^{j-l}\|\|\tau(\cdot,t_l,k)\| \\ &\le C_S\big(\|E_k(\cdot,0)\| + k\sum_{l=1}^{j}\|H^{j-l}\|\|\tau(\cdot,t_l,k)\|\big).\end{aligned}$$

Let the scheme be of order $q \ge 1$, then it is consistent and it follows

$$\|E_k(\cdot,t_j)\| \le C_S\big(\|E_k(\cdot,0)\| + TCk^q\big)$$

The last problem is the initial error. If $\|E_k(\cdot,0)\| = 0$, then the proof is complete. Otherwise we have to assume that for sufficient small k the step size h can be selected sufficiently small. ☐