Outline

General linear partial differential equation of second order

General form:

$$Lu := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x,u) \tag{9}$$

Ingridients:

- $a_{ij}, b_i, c : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable
- wanted function u in n variables, thus $u = u(x_1, \dots, x_n)$
- homogeneous if $f = 0$, inhomogeneous otherwise

General linear partial differential equation of second order

Example

1. *For the three-dimensional Poisson equation* $\Delta u = f$ *holds*

$$A(x) = I \in \mathbb{R}^{3\times 3} \qquad \textit{as} \qquad u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}.$$

2. *For the heat equation* $u_t = \Delta u$ *in 2D with* $u = u(x_1, x_2, t)$ *the operator reads*

$$Lu = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial u}{\partial t}$$

and the coefficients are

$$A = \operatorname{diag}(1,1,0) \in \mathbb{R}^{3\times 3}, \quad b = (0,0,-1)^T.$$

3. *For the 2D wave equation* $u_{tt} = \Delta u$ *holds*

$$Lu = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial t^2} \qquad \textit{and thus} \qquad A = \operatorname{diag}(1,1,-1).$$

Outline

Characteristics of second order partial differential equations

Definition 3.1

For the PDE from (9) the second order term

$$L_0 u := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

is called the principal part of the operator Lu.

Remarks:

- the principal part decides about the main properties of the considered PDE
- the spectrum of the matrix $A(x)$ defines the type of the equation
- without limitation of generality we assume always $A(x) \in \mathbb{R}^{n \times n}$ to be symmetric, otherwise use $\frac{1}{2}(A(x) + A(x)^T)$

Principal symbol

Definition 3.2

The quadratic polynomial

$$L_0(x,\xi) := \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\xi_i\xi_j = \xi^T A(x)\xi$$

is called the <u>*principal symbol*</u> *of a linear second order PDE.*

Example

1. *For the Laplace equation* $\Delta u = 0$ *the principal symbol is*

$$L_0 = \xi_1^2 + \xi_2^2 + \xi_3^2 = \|\xi\|_2^2.$$

2. *For the heat equation* $u_t = \Delta u$ *the principal symbol is*

$$L_0 = \xi_2^2 + \xi_3^2.$$

3. *For the wave equation* $u_{tt} = \Delta u$ *the principal symbol is*

$$L_0 = \xi_1^2 + \xi_2^2 - \xi_3^2.$$

Characteristics

Definition 3.3

The characteristic equation of a given PDE is

$$L_0(x, \xi) = 0.$$

A solution ξ of the characteristic equation is called characteristic direction or characteristic of the principal symbol. A differentiable hypersurface in $\mathbb{R}^n$ such that its normal vector $\xi \in \mathbb{R}^n$ points in a characteristic direction is called a characteristic surface (or characteristic curve if $n = 2$).

Outline

Classification of a second order PDE

Definition 3.4

Let $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* $A(x) \in \mathbb{R}^{n\times n}$ *with respect to their multiplicities. At the point* x *the PDE is called*

(a) *elliptic if all eigenvalues of* $A(x)$ *are positive or all eigenvalues of* $A(x)$ *are negative,*

(b) *parabolic if one eigenvalue is zero and all others are positive or all others are negative,*

(c) *hyperbolic if one eigenvalue is positive and all others are negative or one eigenvalue is negative and all others are positive.*

The PDE is called elliptic, parabolic resp. hyperbolic if it is elliptic, parabolic resp. hyperbolic at every point of the domain Ω.

Classification of a second order PDE

Example

1. *For the* n*th-dimensional Laplace equation the matrix is* $A = I_n$*. Its eigenvalues are* $\lambda_1 = \ldots = \lambda_n = 1$ *and the PDE is elliptic.*
2. *For the 2D heat equation* $u_t = \Delta u$ *with* $u = u(x_1, x_2, t)$ *the matrix is*

   $$A = \mathrm{diag}(1, 1, 0) \in \mathbb{R}^{3\times 3}$$

   and the eigenvalues are $\lambda_1 = \lambda_2 = 1$*,* $\lambda_3 = 0$*. Thereby the PDE is parabolic.*

Classification of a second order PDE

Example

3. *For the 2D wave equation* $u_{tt} = \Delta u$ *the matrix is*

$$A = \operatorname{diag}(1, 1, -1)$$

   and has the eigenvalues $\lambda_1 = \lambda_2 = 1$*,* $\lambda_3 = -1$*. The PDE is hyperbolic.*

4. *Consider the PDE*

$$3u_{xx} + 2u_{xy} + 2u_{xz} + 2u_{yy} + 2u_{zz} = f(x).$$

   The matrix is

$$A(x) = A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

   A check of the main minors shows, that A *is positive definite and thereby all eigenvalues are positive. Indeed they are* $\lambda_1 = 4$*,* $\lambda_2 = 2$*,* $\lambda_3 = 1$*. Thus the PDE is elliptic.*

Classification of a second order PDE

Remark 6

A special case is for $n = 2$*. Using* (x, y) *instead of* (x_1, x_2) *we can write the PDE as*

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + c = f.$$

Without using the eigenvalues directly we can classify the PDE with $a_{11} > 0$ *as*

- *elliptic if* $\det(A) = a_{11}a_{22} - a_{12}^2 > 0$*,*
- *parabolic if* $\det(A) = a_{11}a_{22} - a_{12}^2 = 0$ *and*
- *hyperbolic if* $\det(A) = a_{11}a_{22} - a_{12}^2 < 0$*.*

Outline

Canonical form of a linear second order PDE

Definition 3.5

A linear second order PDE is called of canonical form if it does not include mixed derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ *with* $i \neq j$ *of the unknown function. Thus the PDE is of the form*

$$Lu = \sum_{i=1}^{n} a_{ii}(x) \frac{\partial^2 u}{\partial^2 x_i} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) = f(x).$$

In the special case of just $n = 2$ *coordinates we also call a form*

$$Lu = \frac{\partial^2 u}{\partial x \partial y} + b_1(x,y)u_x + b_2(x,y)u_y + c(x,y) = f(x,y)$$

a canonical form.

Transformation into the canonical form

Symmetric but not diagonal

$$A \in \mathbb{R}^{n\times n}, \qquad A^T = A$$

Diagonalization by eigenvalues and -vectors

$$C^T A C = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

New coordinates

$$\xi \in \mathbb{R}^n, \quad \xi = C^T x, \quad \nabla_x u = C \nabla_\xi u$$

Canonical form

$$\sum_{i=1}^{n} \lambda_i \frac{\partial^2 u}{\partial \xi_i^2} + \sum_{i=1}^{n} \tilde{b}_i \frac{\partial u}{\partial \xi_i} + cu = f$$

Transformation into the canonical form

Example

Consider the PDE $u_{xx} + 2u_{xz} + u_{yy} + 2u_{yz} + 2u_{zz} = 0$*. The matrix reads*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

and its eigenvalues and -vectors are $\lambda_1 = 0$*,* $\lambda_2 = 1$*,* $\lambda_3 = 3$*,*

$$v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

The new coordinates are

$$\xi = \frac{1}{\sqrt{3}}(x + y - z), \quad \eta = \frac{1}{\sqrt{2}}(x - y), \quad \zeta = \frac{1}{2}(x + y + 2z)$$

and the transformed PDE reads

$$u_{\eta\eta} + 3u_{\zeta\zeta} = 0.$$

Transformation into the canonical form for $n = 2$

Consider the PDE

$$Lu = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + \text{lower order terms} = f.$$

Transformation by introducing functionally independent new coordinates

$$\xi = \xi(x,y),\ \eta = \eta(x,y) \quad \text{with} \quad \det(J_{\xi,\eta}(x,y)) = \left| \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \right| \neq 0. \qquad (10)$$

Theorem 3.6

The type of a PDE does not change under a nonsingular coordinate transformation from (10).

Transformation into the canonical form for $n = 2$

Constructing the coordinate system:

- compute the characteristics of the PDE using

$$a_{11}(x,y)w_x^2 + 2a_{12}(x,y)w_x w_y + a_{22}w_y^2 = 0$$

- the solutions of the quadratic equation

$$a_{11}(x,y)\mu^2 + 2a_{12}(x)\mu + a_{22}(x,y) = 0$$

  are

$$\frac{w_x}{w_y} = \mu_{1,2} = \frac{-a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}$$

Three cases:

- elliptic case, $a_{12}^2 - a_{11}a_{22} < 0$, both solutions μ_1 and μ_2 are not real
- parabolic case, $a_{12}^2 - a_{11}a_{22} = 0$, $\mu_1 = \mu_2 = -a_{12}(x,y)/a_{11}(x,y)$
- hyperbolic case, $a_{12}^2 - a_{11}a_{22} > 0$, two different real solutions

Transformation into the canonical form for $n = 2$

Hyperbolic case:

- the transformation is based on the solution of the homogeneous linear first order PDEs

$$w_x - \mu_{1,2}(x,y) w_y = 0$$

- solve the ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mu_{1,2}(x,y) \tag{11}$$

- their solutions result in the new coordinates

$$\xi(x,y) = C_1, \quad \eta(x,y) = C_2$$

- in the new coordinates the PDE has the canonical form

$$u_{\xi\eta} + \text{lower order terms} = f$$

Transformation into the canonical form for $n = 2$

Parabolic case:

- solve the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mu_{1,2}(x,y) = -a_{12}(x,y)/a_{11}(x,y)$$

- the new coordinates are

$$\xi = x, \quad \eta = \eta(x,y)$$

- the transformed PDE has the canonical form

$$u_{\xi\xi} + \text{lower order terms} = f$$

Elliptic case:

- both solutions are complex with $\mu_2 = \overline{\mu}_1$
- with $w(x,y) = c$ being the (complex-valued) solution of the ODE from (11) use

$$\begin{aligned} \xi(x,y) &= w(x,y) + \overline{w(x,y)} = 2\,\mathrm{Re}(w(x,y)), \\ \eta(x,y) &= 2\,\mathrm{Im}(w(x,y)) \end{aligned}$$

- the transformed PDE has the canonical form

$$u_{\xi\xi} + u_{\eta\eta} + \text{lower order terms} = f.$$

Outline

The Laplace equation

Definition 3.7

A function $u \in C^2(\Omega)$ is called harmonic if $\Delta u = 0$ on Ω.

Example

1. *The function $u(x) = x_1 x_2 \cdot \ldots \cdot x_n$ is harmonic on $\mathbb{R}^n$ as also $\Delta u = 0$.*
2. *The function $u(x, y) = \ln(x^2 + y^2)$ is harmonic outside the origin.*
3. *The function $u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ is harmonic outside the origin.*
4. *The fundamental solution of the Laplace equation*

$$E(x) = \begin{cases} -\frac{1}{2\pi} \ln \frac{1}{\|x\|} & \text{for } n = 2, \\ -\frac{1}{(n-2)w_n} \frac{1}{\|x\|^{n-2}} & \text{for } n \geq 3, \end{cases}$$

is harmonic in $\mathbb{R}^n \setminus \{0\}$. Therein w_n is the measure of the unit spehre in $\mathbb{R}^n$ and $\Gamma(z)$ is the Gamma function, both defined as

$$w_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad \Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\, \mathrm{d}t.$$

The Laplace equation

The maximum principle:

- is of fundamental importance in the study of elliptic and parabolic PDEs
- Green's representation theorem and the mean value theorem are needed
- Green's theorem (a special case of Stokes' theorem) allows to express the integral over a plane surface by a curve integral
- the mean vaue theorem says that the value of u at the center equals the average of u over the spehre $S_R(x_0)$ and over the ball $B_R(x_0)$

Theorem 3.8 (Green's representation theorem)

Let u be harmonic on a domain $\Omega \subset \mathbb{R}^n$. For $x_0 \in \Omega$ holds

$$u(x_0) = \int_{\partial\Omega} \left(u(x) \frac{\partial E(x - x_0)}{\partial \overrightarrow{n}} - E(x - x_0) \frac{\partial u(x)}{\partial \overrightarrow{n}} \right) \mathrm{d}\sigma$$

with $\overrightarrow{n} = \overrightarrow{n}(x)$ being the normal derivative at $\partial\Omega$ in the direction of the outside.

The Laplace equation

Theorem 3.9 (Mean value theorem)

Let

$$B_R(x_0) = \{x \in \mathbb{R}^n : \ \|x - x_0\| < R\}$$

be the n*-dimensional ball with radius* R *and center* x_0 *as well as*

$$S_R(x_0) = \{x \in \mathbb{R}^n : \ \|x - x_0\| = R\}$$

its boundary (S *for sphere) and* $\overline{B_R(x_0)} = B_R(x_0) \cup S_R(x_0)$ *be the closure. If* u *is harmonic in* $B_R(x_0)$ *and continuous on* $\overline{B_R(x_0)}$*, then*

$$u(x_0) = \frac{1}{\mathrm{mes}(S_R)} \int_{S_R(x_0)} u(x) \, \mathrm{d}\sigma$$

and

$$u(x_0) = \frac{1}{\mathrm{mes}(B_R)} \int_{B_R(x_0)} u(x) \, \mathrm{d}x.$$

The maximum principle

Theorem 3.10 (Maximum principle)

Let u be harmonic in Ω and continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$.

(a) *Then u attains its maximum and its minimum on the boundary of $\partial\Omega$.*

(b) *If u is not constant, then u attains its maximum and its minimum <u>only</u> on the $\partial\Omega$.*

Remark 7

Part (a) from Theorem 3.10 is called the weak maximum principle and part (b) is called the strong maximum principle.

The maximum principle

Example

Let $\Omega = (-2,2) \times (0,2)$ *and* $u : \mathbb{R}^2 \to \mathbb{R}$ *be the solution of the Cauchy-problem*

$$\begin{aligned} \Delta u &= 0 \quad \text{for } x \in \Omega, \\ u(x) &= |x_1| + 2x_2 \quad \text{for } x \in \partial\Omega. \end{aligned}$$

In orde to compute $m_1 = \max_{x \in \bar{\Omega}} u(x)$ *as well as* $m_2 = \min_{x \in \bar{\Omega}} u(x)$ *we apply the maximum principle and get*

$$\begin{aligned} m_1 &= \max_{x \in \partial\Omega} |x_1| + 2x_2 = 6 \\ m_2 &= \min_{x \in \partial\Omega} |x_1| + 2x_2 = 0. \end{aligned}$$

Further, e. g. locations of the extrema are $\tilde{x} = (-2,2)$ *and* $\bar{x} = (0,0)$ *with* $u(\tilde{x}) = m_1$ *and* $u(\bar{x}) = m_2$*.*

Solution for the Laplace equation on a disc

Domain:

$$B_R = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 < R^2\}$$

Boundary value problem:

$$\begin{aligned} &u_{xx}(x,y) + u_{yy}(x,y) = 0 \quad \text{for } (x,y) \in B, \\ &u(x,y) = g(x,y) \quad \text{for } (x,y) \in \partial B \end{aligned}$$

Work with polar coordinates:

$$x = r\cos(\varphi), \quad y = r\sin(\varphi)$$

Transformed problem:

$$\begin{aligned} &\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0 \quad \text{for } 0 \le r < R,\ 0 \le \varphi < 2\pi, \\ &u(R,\varphi) = g(R\cos(\varphi), R\sin(\varphi)) \quad \text{for } 0 \le \varphi < 2\pi. \end{aligned}$$

Solution for the Laplace equation on a disc

Factorial approach:

$$u(r,\varphi) = F(r)\cdot\Phi(\varphi)$$

Substituting this u in the PDE:

$$F''(r)\Phi(\varphi) + \frac{1}{r}F'(r)\Phi(\varphi) + \frac{1}{r^2}F(r)\Phi''(\varphi) = 0$$

Separation of variables

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)}$$

Both fractions have to be constant

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \mu \qquad \text{with} \qquad \mu = n^2,\ n = 0, 1, 2, \ldots$$

Solution for the Laplace equation on a disc

The solutions of the PDE are

$$u_n(r,\varphi) = r^n\big(c_n\cos(n\varphi) + d_n\sin(n\varphi)\big), \quad n = 0,1,2,\dots$$

The general solution can be written as

$$u(r,\varphi) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n(c_n\cos(n\varphi) + d_n\sin(n\varphi))$$

Theorem 3.11

Let $\tilde{g}(R,\varphi) = g(R\cos(\varphi), R\sin(\varphi))$ *be continuous and* 2π *periodic. Then the coefficients to determine the solution are*

$$c_n = \frac{1}{\pi R^n}\int_0^{2\pi} \tilde{g}(R,\varphi)\cos(n\varphi)\,\mathrm{d}\varphi,$$

$$d_n = \frac{1}{\pi R^n}\int_0^{2\pi} \tilde{g}(R,\varphi)\sin(n\varphi)\,\mathrm{d}\varphi.$$

Solution for the Laplace equation on a disc

Example

Consider with $\Omega = \{x \in \mathbb{R}^2 : \ \|x\|_2 < 2\}$ *the boundary value problem*

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u(x,y) &= |y| \text{ on } \partial\Omega. \end{aligned}$$

With $g(\varphi) = 2|\sin(\varphi)|$ *we get* $d_n = 0$ *for all* n *as well as*

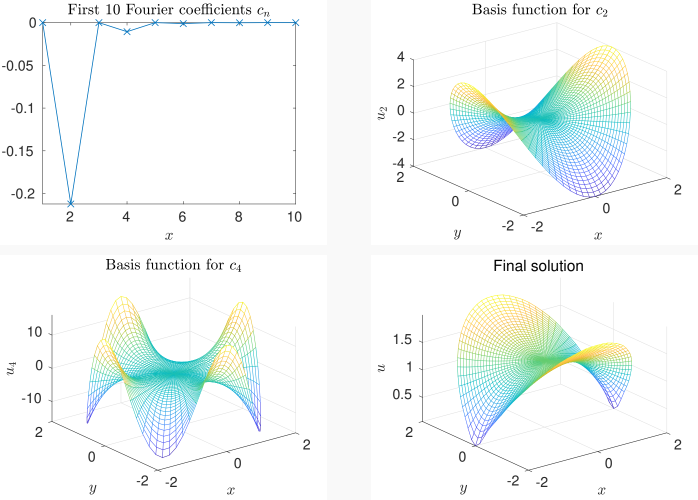
$$c_n = \begin{cases} -\frac{8}{\pi 2^n (n^2-1)} & \text{for even } n, \\ 0 & \text{for odd } n \end{cases}$$

and the solution is

$$u(r,\varphi) = \frac{4}{\pi} - \frac{8}{\pi}\left(\frac{r^2 \cos(2\varphi)}{2^2(2^2-1)} + \frac{r^4 \cos(4\varphi)}{2^4(4^2-1)} + \frac{r^6 \cos(6\varphi)}{2^6(6^2-1)} + \dots \right).$$

Solution for the Laplace equation on a disc

Example (Continuation)



The Dirichlet problem for the Laplace equation on a ball

Theorem 3.12 (Poisson's formula)

Consider the boundary value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{on } B, \\ u &= g \quad \text{on } \partial B. \end{aligned}$$

Its solution has the form

$$u(r,\varphi) = \frac{1}{w_n R} \int_{\partial B} \frac{(R^2 - r^2) g(y)}{\|x-y\|^n} \, \mathrm{d}\sigma_y$$

with w_n being the measure of the unit spehre in $\mathbb{R}^n$.

Outline

The wave equation

General form of the PDE:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{for } x = (x_1, \dots, x_n)^T \in \mathbb{R}^n,\ t > 0$$

Initial conditions:

$$\begin{aligned} u(x,0) &= \phi(x) \quad \text{for } x \in \mathbb{R}^n, \\ u_t(x,0) &= \psi(x) \quad \text{for } x \in \mathbb{R}^n \end{aligned}$$

Problem in 1D:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } -\infty < x < \infty,\ t > 0, \\ &u(x,0) = \phi(x) \quad \text{for } -\infty < x < \infty, \\ &u_t(x,0) = \psi(x) \quad \text{for } -\infty < x < \infty. \end{aligned}$$

Solution of the one-dimensional wave equation

Coordinate transformation ($x+t=C_1$ and $x-t=C_2$ are the characteristic curves of the PDE):

$$\xi = x+t, \quad \eta = x-t$$

Theorem 3.13 (d'Alembert's formula)

For the one-dimensional wave equation with initial conditions $\phi \in C^2(\mathbb{R})$ *and* $\psi \in C^1(\mathbb{R})$ *the unique solution is*

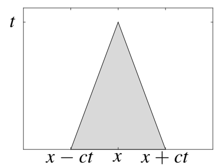
$$u(x,t) = \frac{\phi(x+t)+\phi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(\tau)\,\mathrm{d}\tau.$$

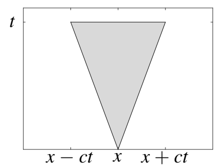
Dependency domain

Remark 8

1. *Dependency domain: the solution $u(x,t)$ at a position x and the time t depends on the initial data ϕ and ψ in the domain $[x-t, x+t]$.*
2. *Domain of influence: the initial data $\phi(x,0)$ and $\psi(x,0)$ at a position x affects the solution at the time t in the range $[x-t, x+t]$.*

   For $u_{tt} - c^2 u_{xx} = 0$ holds



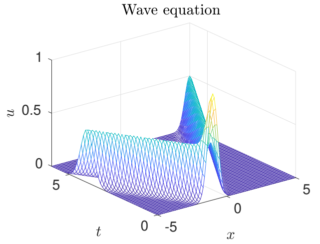


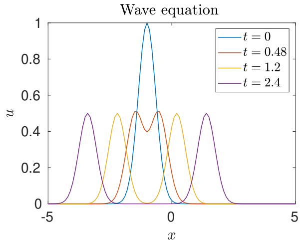
One-dimensional wave equation

Example

Consider the one dimensional wave equation for the initial condition $\phi(x) = \exp(-4(x+1)^2)$ *and* $\psi(x) = 0$. *Following d'Alembert's formula the solution is*

$$u(x,t) = \frac{\exp(-4(x+t+1)^2) + \exp(-4(x-t+1)^2)}{2}.$$



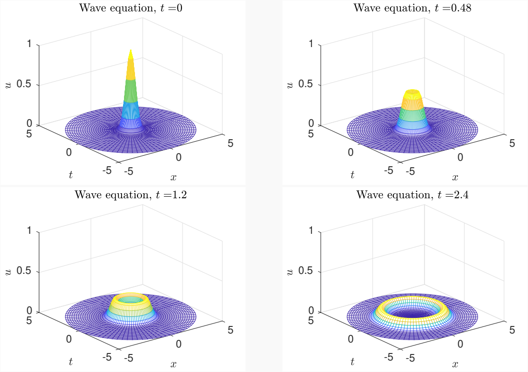


Two-dimensional wave equation

Example

The two dimensional wave equation on a disc for the initial data

$$u(x,0) = \exp(-4(x^2 + y^2)).$$



Reformulation as a system for 1D

Consider $u_{tt} = u_{xx}$ and the quantities

$$z = \begin{pmatrix} v \\ \varepsilon \end{pmatrix} = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$$

Reformulation as a system of two first order PDEs

$$\begin{pmatrix} v_t \\ \varepsilon_t \end{pmatrix} + A \begin{pmatrix} v_x \\ \varepsilon_x \end{pmatrix} = 0 \quad \Rightarrow \quad A(x,t) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Encoupling the system with the new coordinates $w_1 = z_1 + z_2$, $w_2 = z_1 - z_2$ results in

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The initial value problem in the three-dimensional case

Theorem 3.14 (Kirchhoff's formula)

The solution of the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \textit{for } x \in \Omega,\ t > 0$$
$$u(x,0) = 0, \quad u_t(x,0) = p(x) \quad \textit{for } x \in \Omega$$

for $\Omega \subset \mathbb{R}^3$ *and a two times continuously differentiable function* $p : \Omega \to \mathbb{R}$ *is*

$$u(x,t) = u_p(x,t) = \frac{1}{4\pi t} \iint_{S(x,t)} p(y)\, \mathrm{d}y$$

with $S(x,t)$ *the three-dimensional sphere with radius* t *and center* x*, thus* $S(x,t) = \{y \in \mathbb{R}^3 :\ \|y - x\|_2 = t\}$*.*

The initial value problem in the two-dimensional case

Theorem 3.15

The solution of the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \textit{for}\, x \in \Omega,\ t > 0$$
$$u(x,0) = u_0, \quad u_t(x,0) = v_0(x) \quad \textit{for}\, x \in \Omega$$

for $\Omega \subset \mathbb{R}^2$ *and* $u_0, v_0 \in \mathcal{C}^2(\Omega)$ *is*

$$u(x,t) = \frac{1}{2\pi} \iint\limits_{B(x,t)} \frac{v_0(y)}{\sqrt{t^2 - \|y-x\|^2}}\, \mathrm{d}y + \frac{\partial}{\partial t}\left(\iint_{B(x,t)} \frac{u_0(y)}{\sqrt{t^2 - \|y-x\|^2}}\, \mathrm{d}y \right).$$

The initial boundary value problem in the one-dimensional case

Consider the problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0, \quad 0 < x < 1,\ t > 0, \\ &u(0,t) = u(1,t) = 0, \quad t \geq 0, \\ &u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), \quad 0 < x < 1. \end{aligned}$$

Factorial approach and Fourier series result in the following theorem.

Theorem 3.16

The solution reads

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\big(a_n \cos(n\pi t) + b_n \sin(n\pi t)\big) \tag{12}$$

and the coeffcients are

$$a_n = 2\int_0^1 \phi(x)\sin(n\pi x)\,\mathrm{d}x, \qquad b_n = \frac{2}{n\pi}\int_0^1 \psi(x)\sin(n\pi x)\,\mathrm{d}x$$

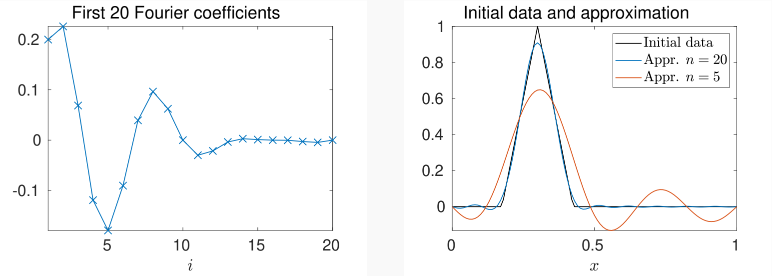
The initial boundary value problem in the one-dimensional case

Example

With $\Omega = (0,1)$ *consider the problem*

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{for } x \in \Omega,\ t > 0, \\ u(0,t) = u(1,t) &= 0 \quad \text{for } t \geq 0, \\ u(x,0) = \max(1 - 8|x - 0.3|, 0), &\quad u_t(x,0) = 0. \end{aligned}$$

We compute the coeffcients for Fourier's series and the final solution for any $t > 0$*.*



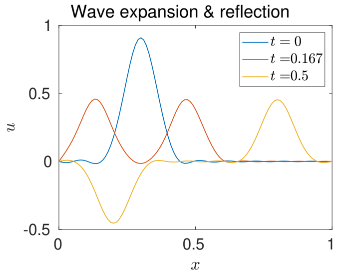
The initial boundary value problem in the one-dimensional case

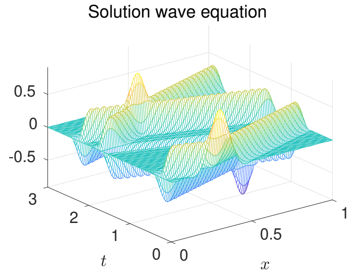
Example

With $\Omega = (0,1)$ consider the problem

$$
\begin{aligned}
&u_{tt} - u_{xx} = 0 \quad \textit{for } x \in \Omega,\ t > 0,\\
&u(0,t) = u(1,t) = 0 \quad \textit{for } t \geq 0,\\
&u(x,0) = \max(1 - 8|x - 0.3|, 0), \quad u_t(x,0) = 0.
\end{aligned}
$$

We compute the coeffcients for Fourier's series and the final solution for any $t > 0$.





Outline

Concentrated energy in one point

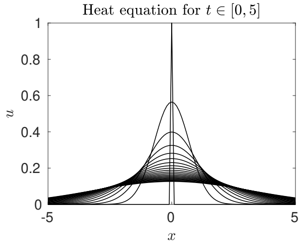
Theorem 3.17

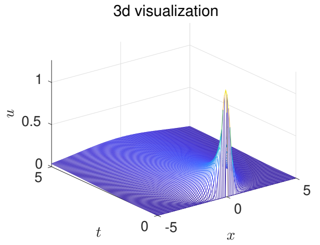
The 1D initial value problem

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \textit{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \begin{cases} \infty & \textit{for } x = 0 \\ 0 & \textit{else} \end{cases} \qquad \textit{for } x \in \mathbb{R}, \end{aligned}$$

has the solution

$$u(x,t) = E(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\Big(-\frac{x^2}{4t}\Big).$$





Concentrated energy in one point

Theorem 3.17

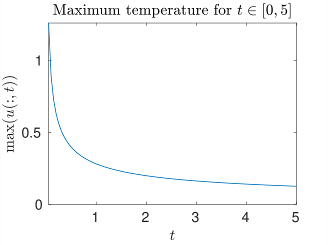
The 1D initial value problem

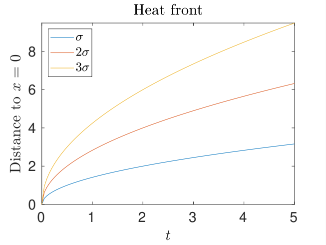
$$u_t - u_{xx} = 0 \quad \textit{for } x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = \begin{cases} \infty & \textit{for } x = 0 \\ 0 & \textit{else} \end{cases} \qquad \textit{for } x \in \mathbb{R},$$

has the solution

$$u(x,t) = E(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(- \frac{x^2}{4t} \right).$$


Maximum temperature for $t \in [0,5]$


Heat front

The initial value problem (Cauchy problem) for the heat equation

Theorem 3.18

Let ϕ be continuous and bounded in $\mathbb{R}^n$. Then the uniquely determined solution of the homogeneous heat equation is

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \phi(y) \exp\Big(-\frac{\|y-x\|_2^2}{4t}\Big)\,\mathrm{d}y \tag{13}$$

with $x, y \in \mathbb{R}^n$ and $t > 0$.

Remark 9

Formula (13) can also be written as

$$u(x,t) = \int_{\mathbb{R}^n} \phi(y) E(y-x,t)\,\mathrm{d}y.$$

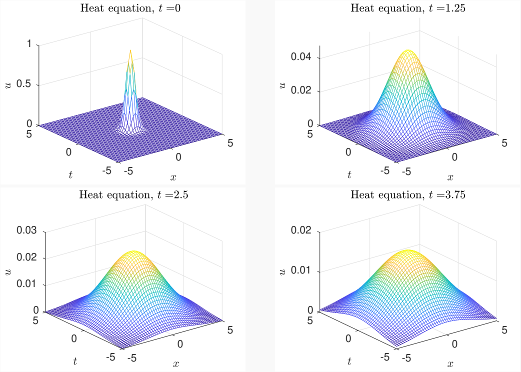
This means $u(x,t)$ is the convolution of the fundamental solution for $y-x$ and ϕ.

Homogeneous heat equation in the two-dimensional case

Example

Consider the two dimensional heat equation for the initial condition

$$\phi(x) = \exp(-4(x_1^2 + x_2^2)).$$



The maximum principle for the heat equation

Theorem 3.19 (Maximum principle)

Let $T > 0$ *and* $u \in C^2(\Omega \times (0,T)) \cap C^0(\overline{\Omega} \times [0,T])$ *be a smooth solution of the homogeneous heat equation*

$$u_t(x,t) = u_{xx}(x,t) \quad \text{for } (x,t) \in \Omega \times (0,T)$$

then

$$\min_{(x,t)\in A} u(x,t) \leq \inf_{(x,t)\in B} u(x,t) \leq \sup_{(x,t)\in B} u(x,t) \leq \max_{(x,t)\in A} u(x,t)$$

with $A = \partial\Omega \times [0,T] \cup \Omega \times \{0\}$ *and* $B = \overline{\Omega} \times [0,T]$.

The initial-boundary value problem for the heat equation

Consider the one-dimensional initial-boundary value problem

$$u_t - u_{xx} = 0 \text{ for } x \in (0,1),\ t > 0, \tag{14}$$

$$u(0,t) = u(1,t) = 0 \text{ for } t > 0, \tag{15}$$

$$u(x,0) = \phi(x) \text{ for } x \in (0,1) \tag{16}$$

Theorem 3.20

Let $\phi \in \mathcal{C}^1([0,1])$ *and compatible condition* $\phi(0) = \phi(1) = 0$*.*

Then the series

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \exp(-n^2\pi^2 t)$$

with the coeffcients

$$c_n = 2\int_0^1 \phi(x) \sin(n\pi x)\,\mathrm{d}x$$

is the uniquely determined solution of (14), (15) and (16).

The initial-boundary value problem for the heat equation

Example

We consider the one-dimensional initial-boundary value heat equation with $\phi(x) = x\sin(\pi x)$*. The formula for computing* c_n *is*

$$c_n = 2\int_0^1 x\sin(\pi x)\sin(n\pi x)\,\mathrm{d}x.$$

The coeffcients are

$$c_1 = \frac{1}{2}, \qquad c_n = \begin{cases} -\frac{8n}{\pi^2(n^2-1)^2} & \textit{for even } n, \\ 0 & \textit{for odd } n \end{cases}$$

Using them the solution reads

$$u(x,t) = \frac{1}{2}\sin(\pi x)\exp(-\pi^2 t) - \sum_{i=1}^{\infty}\frac{16i}{(4i^2-1)^2\pi^2}\sin(2i\pi x)\exp(-4i^2\pi^2 t).$$

Outline

3. Solution of second order partial differential equation

Difference methods for elliptic problems

Idea/approach:

- extension of the 1D approach for boundary value problems with ODEs
- no time dependence, only the static case
- discretization of the bounded domain Ω by a grid with step size h
- approximation of the exact solution in the grid points $U_i \approx u(x^{(i)}$
- substitution of derivatives in the PDE by finite differences, e. g.

$$u_{xx} \approx \frac{1}{h^2}(u(x+h,y) - 2u(x,y) + u(x-h,y))$$

Result:

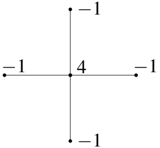
- large systems of linear equations for $U_1, \ldots, U_N$ with N the number of grid points
- quadratic systems $AU = b$, solution by iterative methods

Consider the 2D problem, thus $x \in \mathbb{R}^2$,

$$\begin{aligned} -\Delta u &= f(x) \quad \text{in } \Omega \\ u(x) &= g(x) \quad \text{on } \partial\Omega \end{aligned}$$

5-point stencil from Definition 1.7:

$$L_h^{(5)} = \frac{1}{h^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$



With $h_x = h_y = h$ this results in

$$-\Delta u = -u_{xx} - u_{yy} \approx \frac{-u(x-h,y) - u(x,y-h) + 4u(x,y) - u(x+h,y) - u(x,y+h)}{h^2}$$

Difference methods for elliptic problems

In practice:

- use $x \in \mathbb{R}^2$ instead of (x, y) in order to avoid conflicts in the numbering
- with m nodes in x_1-direction and nodes in x_2-direction we have $N = mn$ grid points

$$x_1, x_2, \ldots, x_N \in \mathbb{R}^2$$

- their associated approximations to $u(x)$ are

$$U_i \approx u(x_i)$$

5-point stencil:

- grid point $x_{(j-1)m+i}$ has connections to the neighbours

$$x_{(j-2)m+i},\ x_{jm+i},\ x_{(j-1)m+i-1},\ x_{(j-1)m+i+1}$$

- consequently a discretization of $-\Delta u$ with $h_x = h_y$ is

$$-\frac{U_{(j-2)m+i} + U_{jm+i} - 4U_{(j-1)m+i} + U_{(j-1)m+i-1} + U_{(j-1)m+i+1}}{h^2}$$

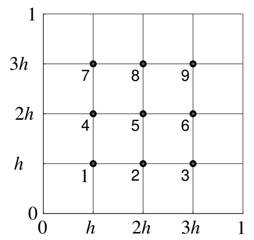
Difference methods for elliptic problems

Example

Consider for $\Omega = (0,1)^2$ *the boundary value problem*

$$-\Delta u = 7, \quad \textit{in } \Omega, \qquad u(x,y) = g(x,y), \quad \textit{on } \partial\Omega.$$

For $m = 3$ *and* $h = 1/(m+1)$ *the grid is*



Difference methods for elliptic problems

Example

Consider for $\Omega = (0,1)^2$ *the boundary value problem*

$$-\Delta u = 7, \quad \text{in } \Omega, \qquad u(x,y) = g(x,y), \quad \text{on } \partial\Omega.$$

Next we derive the equations for the single points using the PDE $-\Delta u = 7$*.*

Point 1

$$\frac{1}{h^2}(-\underbrace{u(h,0)}_{g(h,0)} - \underbrace{u(0,h)}_{g(h,0)} + 4U_1 - U_2 - U_4) = 7$$

$$\Rightarrow \quad \frac{1}{h^2}(4U_1 - U_2 - U_4) = 7 + \frac{1}{h^2}(g(h,0) + g(0,h)).$$

Difference methods for elliptic problems

Example

Consider for $\Omega = (0,1)^2$ *the boundary value problem*

$$-\Delta u = 7, \quad \text{in } \Omega, \qquad u(x,y) = g(x,y), \quad \text{on } \partial\Omega.$$

Next we derive the equations for the single points using the PDE $-\Delta u = 7$*.*

Point 2

$$\frac{1}{h^2}(-U_1 - u(2h,0) + 4U_2 - U_3 - U_5) = 7$$
$$\Rightarrow \quad \frac{1}{h^2}(-U_1 + 4U_2 - U_3 - U_5) = 7 + \frac{1}{h^2} g(2h,0).$$

Difference methods for elliptic problems

Example

Consider for $\Omega = (0,1)^2$ *the boundary value problem*

$$-\Delta u = 7, \quad \textit{in}\ \Omega, \qquad u(x,y) = g(x,y), \quad \textit{on}\ \partial\Omega.$$

Next we derive the equations for the single points using the PDE $-\Delta u = 7$*.*

Point 5

$$\frac{1}{h^2}(-U_2 - U_4 + 4U_5 - U_6 - U_8) = 7.$$

Difference methods for elliptic problems

Example

Consider for $\Omega = (0,1)^2$ *the boundary value problem*

$$-\Delta u = 7, \quad \text{in } \Omega, \qquad u(x,y) = g(x,y), \quad \text{on } \partial\Omega.$$

The system of linear equations is

$$A \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_9 \end{pmatrix} = \frac{1}{h^2} \begin{pmatrix} 7h^2 + g(0,h) + g(h,0) \\ 7h^2 + g(2h,0) \\ 7h^2 + g(3h,0) + g(1,h) \\ 7h^2 + g(0,2h) \\ 7h^2 \\ 7h^2 + g(1,2h) \\ 7h^2 + g(0,3h) + g(h,1) \\ 7h^2 + g(2h,1) \\ 7h^2 + g(3h,1) + g(1,3h) \end{pmatrix}.$$

Difference methods for elliptic problems

Example

Consider for $\Omega = (0,1)^2$ the boundary value problem

$$-\Delta u = 7, \quad \text{in } \Omega, \qquad u(x,y) = g(x,y), \quad \text{on } \partial\Omega.$$

The discretization matrix reads

$$A = \frac{1}{h^2}\left(\begin{array}{ccc|ccc|ccc}
4 & -1 & & -1 & & & & & \\
-1 & 4 & -1 & & -1 & & & & \\
 & -1 & 4 & & & -1 & & & \\
\hline
-1 & & & 4 & -1 & & -1 & & \\
 & -1 & & -1 & 4 & -1 & & -1 & \\
 & & -1 & & -1 & 4 & & & -1 \\
\hline
 & & & -1 & & & 4 & -1 & \\
 & & & & -1 & & -1 & 4 & -1 \\
 & & & & & -1 & & -1 & 4
\end{array}\right).$$

Difference methods for elliptic problems

Kronecker-product of $U \in \mathbb{R}^{m\times n}$ and $V \in \mathbb{R}^{p\times q}$:

$$U \otimes V = \begin{pmatrix} u_{11}V & \cdots & u_{1n}V \\ \vdots & \ddots & \vdots \\ u_{m1}V & \cdots & u_{mn}V \end{pmatrix} \in \mathbb{R}^{mp\times nq}$$

Representation of the discretization matrix:

$$A = \frac{1}{h^2}\left(I_m \otimes T_m + T_m \otimes I_m \in \mathbb{R}^{m\times m}\right),$$

with $T_m = \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m},\ I_m \in \mathbb{R}^{m\times m}$

Difference methods for elliptic problems

Remark 10

1. *The system of linear equations comprizes only equations for values at inner nodes.*
2. *The structure of the matrix* A *depends on the enumeration of the grid points.*
3. *The right-hand side of the system at grid points* x_ℓ*, which are not connected to the boundary, includes only the values of* $f(x_\ell) = b_\ell$*. Points close to the boundary, i.e. points which have a stencil that contains at least one boundary point, feature a contribution from the corresponding Dirichlet values to the right-hand side* b *of the system.*

Difference methods for elliptic problems

Definition 3.21

A regular matrix $A \in \mathbb{R}^{n\times n}$ *is called an* <u>*M-matrix*</u>*, if*

$$A_{ij} \le 0 \quad \text{for } i \ne j, \qquad\qquad \sum_{j=1}^{n} A_{ij} \ge 0 \quad \text{for } i = 1, \ldots, n,$$

and all entries of A^{-1} *are nonnegative.*

Difference to the analytical solutions:

- due to the construction of the difference approximation holds

$$D_h = L_h^{(5)} u(x) = Lu(x) + \mathcal{O}(h^2)$$

- with $u \in C^4(\Omega) \cap C(\bar{\Omega})$ the local error for D_h is controlled by the 4. derivatives of u

$$L_h^{(5)} u(x) = f(x) + E(x,h), \qquad L_h^{(5)} U(x) = f(x), \qquad \|E(x,h)\| \le \frac{h^2}{12} \max_{x\in\Omega} \left(|u^{(4)}(x)|\right)$$

- consistency and stability result in

$$\|u_h - U\|_\infty \le Ch^2,$$

- the constant C depends on the operator and the domain

Difference methods for elliptic problems

Summary of the procedure:

1. discretize the domain with the step size h resp. different step sizes $h_{x_1},\ h_{x_2},\dots$
2. number the grid points $x^{(1)},\dots,x^{(N)}$
3. derive a proper difference stencil for the PDE
4. set up equations for the unknown U_i
5. include boundary conditions in the equations
6. solve the system of linear equations

Difference methods for elliptic problems

Example

Consider for $\Omega = (0,1)^2$ *the boundary value problem*

$$\begin{aligned} -\Delta u + a u_{x_1} + b u_{x_2} + cu = f &\quad \textit{in } \Omega, \\ u = g &\quad \textit{on } \partial\Omega. \end{aligned}$$

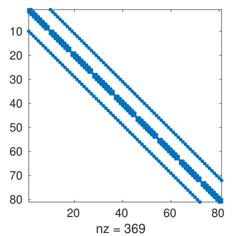
Using $h_x = h_y = h$ *the stencil reads*

$$= \frac{1}{h^2} \begin{bmatrix} & -1 + \frac{h}{2} b_i & \\ -1 - \frac{h}{2} a_i & 4 + h^2 c_i & -1 + \frac{h}{2} a_i \\ & -1 - \frac{h}{2} b_i & \end{bmatrix}.$$

Difference methods for elliptic problems

Example (Continuation)

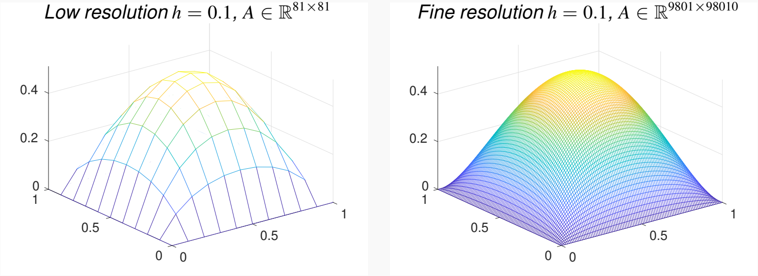
For $m = 9$ *and* $h = 0.1$ *with* $a = b = c = 0$ *the structure of the matrix is*



Only 369 *of* 6561 *matrix elements are nonzero* (5.6%)*. The system matrix is weakly diagonally dominant and irreducible. Therefore use iterative methods to solve the system of linear equations instead of the Gaussian elimination.*

Example (Continuation)

Solution with $f(x) = 12$ *and for a grid with* $h = 0.1$ *resp.* $h = 0.01$*.*



Difference methods for elliptic problems

Example

Next we derive a difference stencil for the PDE

$$-u_{xx} - u_{xy} - u_{yy} + u_x + u_y + u = 0.$$

With $h_x = h_y = h$ *the single parts of the stencil are*

$$-\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -2 & 1 \\ & 1 & \end{bmatrix} - \frac{1}{4h^2}\begin{bmatrix} -1 & & 1 \\ & & \\ 1 & & -1 \end{bmatrix} + \frac{1}{2h}\begin{bmatrix} & & \\ -1 & & 1 \\ & & \end{bmatrix} + \frac{1}{2h}\begin{bmatrix} & 1 & \\ & & \\ & -1 & \end{bmatrix} + \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}$$

Thus the stencil reads

$$\frac{1}{h^2}\begin{bmatrix} \frac{1}{4} & -1 + \frac{h}{2} & -\frac{1}{4} \\ -1 - \frac{h}{2} & 4 + h & -1 + \frac{h}{2} \\ -\frac{1}{4} & -1 - \frac{h}{2} & \frac{1}{4} \end{bmatrix}$$

Include Neumann boundary conditions

Neumann boundary conditions:

$$\frac{\partial u}{\partial \overrightarrow{n}}(x) = \alpha \text{ for } x \in \partial\Omega$$

One-dimensional:

$$\frac{U_n - U_{n-1}}{h} = \alpha \quad \Leftrightarrow \quad U_n - U_{n-1} = h\alpha \text{ with } U_n \approx u(b)$$

Two-dimensional:

- extend the numerb of nodes for boundary point with Neumann conditions

$$\frac{U_{i(m+1)} - U_{i(m+1)-1}}{h} = w(ih) \quad \Leftrightarrow \quad U_{i(m+1)} - U_{i(m+1)-1} = hw(ih)$$

- or use the connection directly for the stencil as

$$u(b, ih) = U_{i(m+1)-1} + hw(ih), \quad i = 1, \ldots, n$$

Irreducible matrices

Definition 3.22

For a (possibly non-symmetric) matrix $A \in \mathbb{R}^{n\times n}$ we call $G_A = (V, E)$ the adjacency graph with the set of nodes $V = \{1, \dots, n\}$ and the set of edges

$$E = \{(i,j) \in V^2 : \; a_{ij} \neq 0\}.$$

Definition 3.23

A matrix A is called irreducible if every node i is connected to every node j by a chain of edges. In the case of an non-symmetric matrix of a chain of equally oriented edges in both directions.

Important also for:

- Perron-Frobenius theory of nonnegative matrices
- Graph theory
- via Perron-Frobenius indirectly for Page-rank, Markov chains, etc.

Irreducible matrices

Theorem 3.24

Let $A \in \mathbb{R}^{n\times n}$ be irreducible, weakly diagonally dominant, thus

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| \leq a_{ii}$$

and let for at least one j the inequality be strict $<$.

Then the Jacobean as well as the Gauss-Seidel iteration converge to the solution of the system of linear equations $Ax = b$ for each initial guess to the solution x^*.

Irreducible matrices

Theorem 3.25

Let $A \in \mathbb{R}^{n \times n}$ be irreducible with

$$a_{ij} \leq 0 \quad \text{for } i \neq j, \qquad \sum_{j=1}^{n} a_{ij} \geq 0 \quad \text{for } i = 1, \ldots, n,$$

as well as $\sum_{j=1}^{n} a_{\ell j} > 0$ for at least one ℓ.

Then

1. A *is inverse positive, that means $A^{-1} \geq 0$ holds component wise,*
2. A *is an M-matrix,*
3. *the Jacobean method to solve $AU = b$ converges,*
4. A *is inverse monotone, that means*

$$Ax \leq Ay \quad \Rightarrow \quad x \leq y.$$

Difference methods for elliptic problems

Theorem 3.26

Let the assumptions from Theorem 3.25 be fulfilled.

It holds:

1. *If for the boundary value problem holds* $L_hU \geq 0$ *in* Ω_h *and* $U \geq 0$ *on* $\partial\Omega_h$*, then holds* $U \geq 0$ *on* $\bar{\Omega}_h$*.*
2. *For the Laplace operator* $L = -\Delta = f$ *on* $\Omega = (0,1)^2$ *and a four times continuously differentiable solution* u *of the boundary value problem holds*

$$\|U(x) - u(x)\|_{\bar{\Omega}_h} \leq Ch^2 \max(|u^{(4)}(x)|).$$

Difference methods for elliptic problems

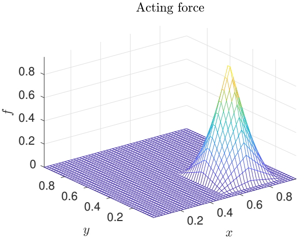
Example

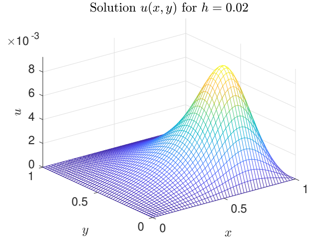
Consider the boundary value problem

$$-\Delta u(x) = f(x) \quad \textit{in } \Omega = (0,1)^2, \qquad u(x) = 0 \quad \textit{on } \partial\Omega$$

with $f(x) = \max\big(0, 1 - 8|x_1 - 0.5|\big) \cdot \max\big(0, 1 - 8|x_2 - 0.5|\big)$.

The acting force and the solution are





Difference methods for elliptic problems

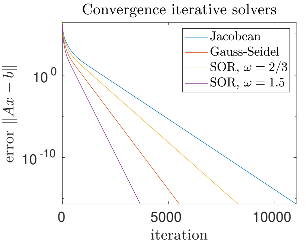
Example

Consider the boundary value problem

$$-\Delta u(x) = f(x) \quad \textit{in}\ \Omega = (0,1)^2, \qquad u(x) = 0 \quad \textit{on}\ \partial\Omega$$

with $f(x) = \max\big(0, 1 - 8|x_1 - 0.5|\big) \cdot \max\big(0, 1 - 8|x_2 - 0.5|\big)$.

Convergence speed for Jacobean, Gauss-Seidel and SOR with $w \in \{\frac{2}{3}, \frac{3}{2}\}$



Keep in mind, that the cg-method converges much faster!

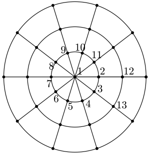
Difference methods on a circ

Grid for a circ:

- use a discretization for the rectangle $(0, R] \times [0, 2\pi)$ by m resp. n nodes with

$$r = (h, 2h, \ldots, mh), \quad \varphi = (0, k, 2k, \ldots, 2\pi - k)$$

- one additional grid point (1st) is the origin
- polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ result in



E. g. the transformed Laplace operator reads:

$$\Delta u(x, y) = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}$$

Outline

Difference methods for parabolic problems

Consider the 1D heat equation for an unknown function $u = u(x,t)$

$$\begin{aligned} u_t - u_{xx} &= f(x,t) \quad \text{in } (0,1) \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{for } x \in [0,1], \\ u(0,t) &= l(t), \\ u(1,t) &= r(t) \end{aligned}$$

Given:

- right-hand side f, heat sources and sinks
- continuously (differentiable) functions u_0, l and r as initial temperature for $t = 0$ and the temperatures at the boundaries $x = 0$ and $x = 1$

Difference methods for parabolic problems

Discretization:

- domain $(0,1) \times (0,T)$ is discretized by the grid

$$\begin{aligned} x_i &= ih, \quad i = 0, \dots, N \\ t_j &= jk, \quad j = 0, \dots, M \end{aligned}$$

- compute approximations

$$U_i^j \approx u(x_i, t_j).$$

Semi-discretization:

- $u(x,0)$ is given, successive calculation for later time layers

$$U_{1:N}^0 = u(x,0) \to U_{1:N}^1 \approx u(x,k) \to \dots \to U_{1:N}^M \approx u(x,T)$$

- transition from time layer $U_{1:N}^j$ to time layer $U_{1:N}^{j+1}$ using single step methods
  (Euler, Crank-Nicolson, etc.)

Difference methods for parabolic problems

Discretization of the time derivative by two-point approximation in t-direction:

$$u_t(x_i, t_j) \approx \frac{U_i^{j+1} - U_i^j}{k}.$$

Discretization in x-direction:

- use the average over both involved time layers

$$u_{xx}(x_i, t_j) \approx (1-\sigma)\frac{U_{i-1}^j - 2U_i^j + U_{i+1}^j}{h^2} + \sigma\frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{h^2}$$

- $\sigma \in [0, 1]$ as a shift for an explicit ($\sigma = 0$) or an implicit ($\sigma > 0$) method

Initial and boundary conditions:

$$\begin{aligned} &U_i^0 = u_0(x_i), \qquad i = 0, \dots, N, \\ &U_0^j = l(t_j), \quad U_N^j = r(t_j), \quad j = 0, \dots, M \end{aligned}$$

Euler explicit for parabolic problems

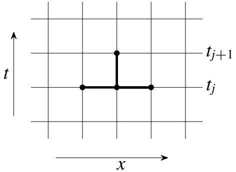
Case $\sigma = 0$:

- Euler-method in time

$$U_i^{j+1} = U_i^j + \frac{k}{h^2}(U_{i-1}^j - 2U_i^j + U_{i+1}^j) + kf_i^j$$

  with $\gamma = \frac{k}{h^2}$
- starting with U_i^j, $i = 0, \ldots, N$, of the time layer j, the explicit scheme enables to compute approximations U_i^{j+1}, $i = 0, \ldots, N$, for the new time layer $(j+1)$

The stencil:



Associated matrix:

$$A = \begin{pmatrix} 1-2\gamma & \gamma & & \\ \gamma & 1-2\gamma & \gamma & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

Crank-Nicolson for parabolic problems

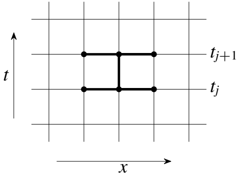
Case $\sigma = 0.5$:

- Crank-Nicolson-method in time

$$2(1+\gamma)U_i^{j+1} - \gamma(U_{i-1}^{j+1} + U_{i+1}^{j+1}) = 2(1-\gamma)U_i^j + \gamma(U_{i-1}^j + U_{i+1}^j) + 2kf(x_i, t_j + 0.5k)$$

- in each layer one has to solve a system of linear equations with a tridiagonal matrix to compute the new approximations U_i^{j+1}, $i = 0, \ldots, N$

The stencil:



Crank-Nicolson for parabolic problems

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- in each layer one has to solve a system of linear equations with a tridiagonal matrix to compute the new approximations U_i^{j+1}, $i = 0, \dots, N$

System of equations:

$$\begin{pmatrix} 2(1+\gamma) & -\gamma & & \\ -\gamma & \ddots & \ddots & \\ & \ddots & \ddots & -\gamma \\ & & -\gamma & 2(1+\gamma) \end{pmatrix} U_:^{j+1} = \begin{pmatrix} 2(1-\gamma) & \gamma & & \\ \gamma & \ddots & \ddots & \\ & \ddots & \ddots & -\gamma \\ & & -\gamma & 2(1-\gamma) \end{pmatrix} U_:^j + 2kf_:^{j+\frac{1}{2}}$$

Euler implicit for parabolic problems

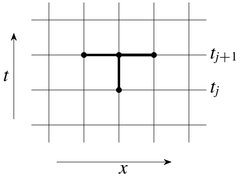
Case $\sigma = 1$:

- implicit Euler-method in time direction

$$(1+2\gamma)U_i^{j+1} - \gamma(U_{i-1}^{j+1} + U_{i+1}^{j+1}) = U_i^j + kf_i^{j+1}$$

- a system of linear equations has to be solved in each time layer, too

The stencil:



Euler implicit for parabolic problems

Case $\sigma = 1$:

- implicit Euler-method in time direction

$$(1+2\gamma)U_i^{j+1} - \gamma(U_{i-1}^{j+1} + U_{i+1}^{j+1}) = U_i^j + kf_i^{j+1}$$

- a system of linear equations has to be solved in each time layer, too

System of equations:

$$\begin{pmatrix} 1+2\gamma & -\gamma & & \\ -\gamma & \ddots & \ddots & \\ & \ddots & \ddots & -\gamma \\ & & -\gamma & 1+2\gamma \end{pmatrix} U_:^{j+1} = U_:^j + kf_:^{j+1}$$

Difference methods for parabolic problems

Local truncation error/consistency:

$\mathcal{O}(h^2 + k)$ for Euler's methods, thus the orders are $2/1$

$\mathcal{O}(h^2 + k^2)$ for Crank-Nicolson, thus the orders are $2/2$

Theorem 3.27 (Stability)

For the solution of the discretized boundary value problem

$$\frac{U_i^{j+1} - U_i^j}{k} - D^{(2)}\left(\sigma U_i^{j+1} + (1-\sigma)U_i^j\right) = f_i^j$$

holds

$$\max_{i,j} |U_i^j| \leq \max_{0\leq x\leq 1} |u_0(x)| + k \sum_{j=0}^{M} \max_{i=1,\dots,N} |f_i^j|,$$

if

$$0 \leq \sigma \leq 1 \quad \textit{and} \quad 1 - 2(1-\sigma)\frac{k}{h^2} \geq 0.$$

Difference methods for parabolic problems

Numerical stability:

- limiting of time step size k depending on step size in space direction h

$$(1-\sigma)\frac{k}{h^2} \leq \frac{1}{2}$$

- Euler explicit ($\sigma = 0$):

$$k \leq \frac{h^2}{2},$$

- Crank-Nicolson ($\sigma = 0.5$):

$$k \leq h^2,$$

- Euler implicit ($\sigma = 1$):

No restriction!

Difference methods for parabolic problems

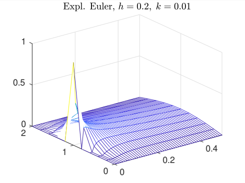
Example

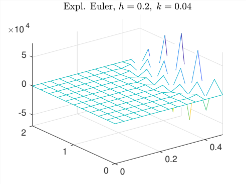
Consider the parabolic initial boundary value problem

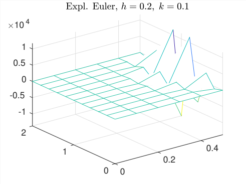
$$u_t = u_{xx} + 0.1\sin(0.5\pi x) \quad \textit{in } \Omega = (0,2),\ t > 0,$$
$$u(x,0) = \max(1 - 5|1-x|, 0), \quad u(0,t) = u(2,t) = 0.$$

Computation of a solution by explicit Euler, Crank-Nicolson and implicit Euler for various step sizes in time direction. With respect to stability we get

$$h = 0.2 \quad \Rightarrow k_{\mathit{Euler}} \leq 0.02, \quad k_{\mathit{CN}} \leq 0.04.$$







Difference methods for parabolic problems

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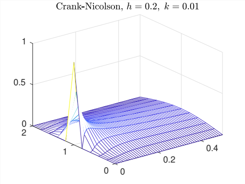
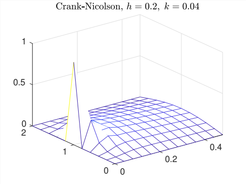
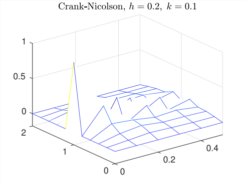

Crank-Nicolson, $h = 0.2,\ k = 0.01$


Crank-Nicolson, $h = 0.2,\ k = 0.04$


Crank-Nicolson, $h = 0.2,\ k = 0.1$

Difference methods for parabolic problems

Example

Consider the parabolic initial boundary value problem

$$u_t = u_{xx} + 0.1\sin(0.5\pi x) \quad \textit{in}\ \Omega = (0,2),\ t > 0,$$
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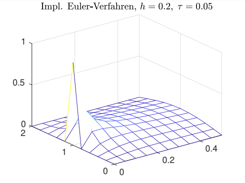
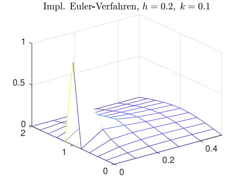

Impl. Euler-Verfahren, $h = 0.2,\ \tau = 0.05$


Impl. Euler-Verfahren, $h = 0.2,\ k = 0.1$

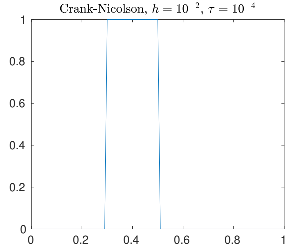
Difference methods for parabolic problems

Example

Consider the initial-boundary value problem

$$u_t = u_{xx}, \qquad (x,y) \in \Omega = (0,1),$$

$$u(x,0) = \begin{cases} 1 & \textit{for } x \in [0.3, 0.5] \\ 0 & \textit{otherwise} \end{cases}, \qquad u(0,t) = u(1,t) = 0.$$



Crank-Nicolson, $h = 10^{-2}$, $\tau = 10^{-4}$

Difference methods for parabolic problems

Example

Numerical solution of 2D heat equation

$$u_t = 0.1\Delta u.$$

We use the implicit Euler method. This means

$$\frac{U(:, i+1) - U(:, i)}{k} = 0.1 L_h^{(5)} U(:, i+1)$$

and thus

$$U(:, i+1) = \left(I_{n^2,n^2} - 0.1kA\right)^{-1} U(:, i), \quad i = 0, 1, 2, \dots .$$

The system matrix is defined using

$$A = \frac{1}{h^2}\left(T \otimes I_{n,n} + I_{n,n} \otimes T\right) \quad \textit{with} \quad T = \operatorname{tridiag}(1, -2, 1) \in \mathbb{R}^{n\times n}.$$

In MatLab use $\operatorname{kron}(T, I_{n,n})$ *for* $T \otimes I_{n,n}$.

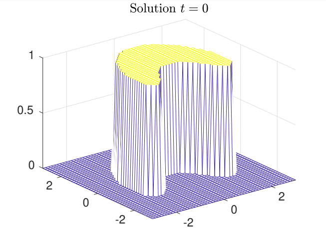
Difference methods for parabolic problems

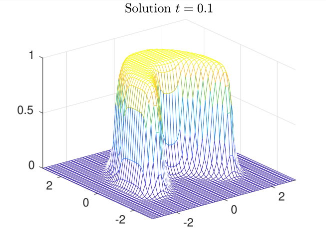
Example

Numerical solution of 2D heat equation

$$u_t = 0.1\Delta u.$$

Some intermediary plots.





Difference methods for parabolic problems

Example

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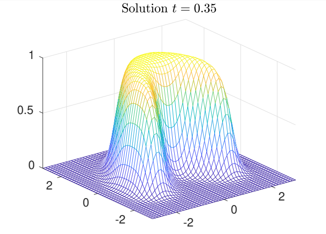
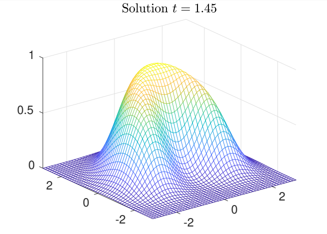


Solution $t = 0.35$



Solution $t = 1.45$

Outline

Difference methods for hyperbolic problems

Standard 1D hyperbolic initial-boundary value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad \text{in } [0,L] \times [0,\infty), \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \\ u(0,t) &= h_1(t), \quad u(L,t) = h_2(t) \end{aligned}$$

Solution in $\mathbb{R} \times [0,\infty)$:

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$$

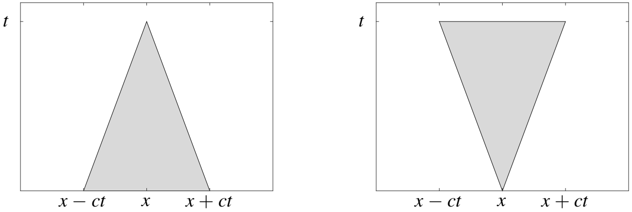
Difference methods for hyperbolic problems

Dependency domain for (x,t):

$$B(x,t) = \{(\xi, t-k)\ :\ k \geq 0,\ |x-\xi| \leq ck\}$$

Domain of influence:

$$E(x,t) = \{(\xi, t+k)\ :\ k \geq 0,\ |x-\xi| \leq ck\}$$



Difference methods for hyperbolic problems

Discretization of u_{xx} in the current time layer (explicit scheme):

$$\frac{1}{k^2}\left(U_i^{j+1} - 2U_i^j + U_i^{j-1}\right) = \frac{c^2}{h^2}\left(U_{i+1}^j - 2U_i^j + U_{i-1}^j\right)$$

Explicit scheme

$$\begin{aligned} U_i^{j+1} &= 2U_i^j - U_i^{j-1} + \frac{c^2k^2}{h^2}\left(U_{i+1}^j - 2U_i^j + U_{i-1}^j\right) \\ &= \rho^2 U_{i-1}^j + 2(1-\rho^2)U_i^j + \rho^2 U_{i+1}^j - U_i^{j-1} \qquad \text{with } \rho = \frac{ck}{h} \end{aligned}$$

The error behaves as:

$$\mathcal{O}(h^2 + k^2)$$

Initial conditions (displacement):

$$U_i^0 = f(x_i)$$

Difference methods for hyperbolic problems

Initial speed (only order 1 in time):

$$\frac{U_i^1 - U_i^0}{k} = g(x_i) \quad \Rightarrow \quad U_i^1 = kg(x_i) + U_i^0$$

Instead use (order 2 in time):

$$\begin{aligned} u(x,t_1) &= u(x,t_0) + ku_t(x,t_0) + \frac{k^2}{2}u_{tt}(x,t_0) \\ &= U_i^0 + kg(x_i) + \frac{c^2k^2}{2}u_{xx}(x_i,t_0) = U_i^0 + kg(x_i) + \frac{c^2k^2}{2}f''(x_i) \\ \Rightarrow \quad U_i^1 &= U_i^0 + kg(x_i) + \frac{c^2k^2}{2}f''(x_i) \end{aligned}$$

Boundary conditions:

$$U_0^j = h_1(t_j), \qquad U_{L/h}^j = h_2(t_j)$$

Difference methods for hyperbolic problems

General scheme (with $0 \leq \sigma \leq \frac{1}{2}$):

$$\begin{aligned} u_{xx} \approx\ & h^{-2}\sigma\left(U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}\right) && \text{layer } j+1 \\ & + h^{-2}(1-2\sigma)\left(U_{i-1}^{j} - 2U_i^{j} + U_{i+1}^{j}\right) && \text{layer } j \\ & + h^{-2}\sigma\left(U_{i-1}^{j-1} - 2U_i^{j-1} + U_{i+1}^{j-1}\right) && \text{layer } j-1 \end{aligned}$$

Numerical stability:

- similar to parabolic problems the time step size is limited by the step size in space direction due to the dependence area
- for explicit Euler this is the so called CFL-condition (Courant-Friedrichs-Levy)

$$ck \leq h$$

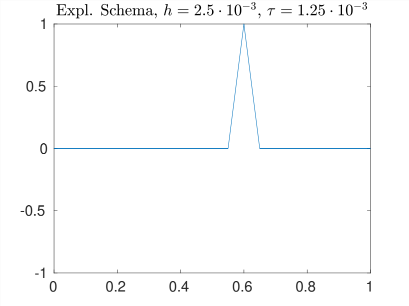
- otherwise the approximations may diverge, although the solution is bounded

Difference methods for hyperbolic problems

Example

Consider the initial-boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad (x,y) \in \Omega = (0,1), \\ u_t(x,0) &= 0, \qquad u(0,t) = u(1,t) = 0. \end{aligned}$$



Difference methods for hyperbolic problems

Example (Oscillation of a square membrane without external influences)

Boundary value problem for $\Omega = (0,1)^2$*:*

$$\begin{aligned} &u_{tt}(x,y,t) = \Delta u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) \qquad \textit{for } (x,y) \in \Omega, \\ &u(x,y) = 0 \qquad \textit{for } (x,y) \in \partial\Omega. \end{aligned}$$

Again the same approach as in 1D $u(x,y,t) = v(x,y)w(t)$ *results in*

$$\begin{aligned} &u_{tt} = vw'', \qquad \Delta u = w\Delta v \\ &w''v = \Delta v w \\ &\frac{\Delta v}{v} = \frac{w''}{w} =: -\lambda. \end{aligned}$$

Difference methods for hyperbolic problems

Example (Continuation)

First problem: computation of the eigenmodes in 2D

$$\begin{aligned} \Delta v &= -\lambda v \qquad \textit{in } \Omega, \\ v &= 0 \qquad \textit{on } \partial\Omega. \end{aligned}$$

Second problem: due to $w'' = -\lambda w$ *the oscillation is*

$$w(t) = a\cos\left(\sqrt{\lambda}t\right) + b\sin\left(\sqrt{\lambda}t\right).$$

λ *with a small amount: low-frequency (fundamental) natural oscillations,*
λ *with a large amount; high-frequency natural oscillations.*

Difference methods for hyperbolic problems

Example (Continuation)

The eigenvalues for the analytical problem are

$$\lambda = -(k\pi)^2 - (\ell\pi)^2, \qquad k, \ell = 1, 2, 3, \ldots$$

and thus

$$\begin{aligned}
(k,\ell) = (1,1) \quad &\Rightarrow \quad \lambda_1 = -2\pi^2, \text{ multiplicity 1} \\
(k,\ell) = (1,2) \quad &\Rightarrow \quad \lambda_{2,3} = -5\pi^2, \text{ mult. 2 as } (k,\ell) = (2,1) \\
(k,\ell) = (2,2) \quad &\Rightarrow \quad \lambda_4 = -8\pi^2, \text{ mult. 1} \\
(k,\ell) = (1,3) \quad &\Rightarrow \quad \lambda_{5,6} = -10\pi^2, \text{ mult. 2 as } (k,\ell) = (3,1)
\end{aligned}$$

Difference methods for hyperbolic problems

Example (Continuation)

Discretization matrix for $n = 3$

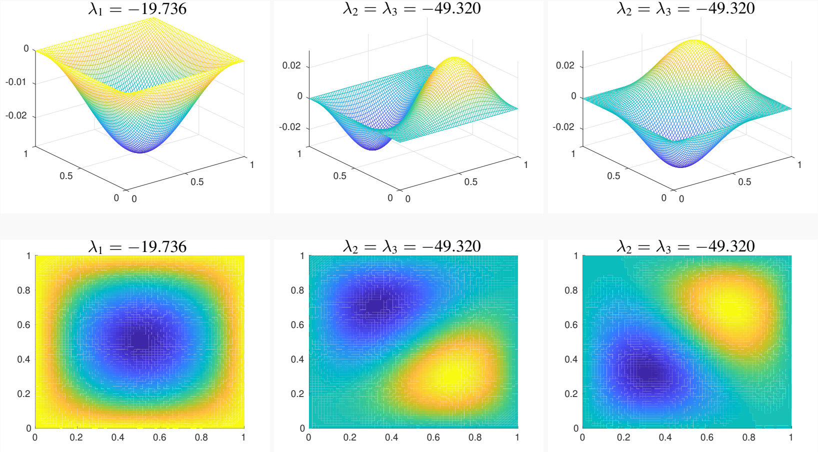
$$A = \frac{1}{h^2} \left(\begin{array}{ccc:ccc:ccc} 4 & -1 & & -1 & & & & & \\ -1 & 4 & -1 & & -1 & & & & \\ & -1 & 4 & & & -1 & & & \\ \hdashline -1 & & & 4 & -1 & & -1 & & \\ & -1 & & -1 & 4 & -1 & & -1 & \\ & & -1 & & -1 & 4 & & & -1 \\ \hdashline & & & -1 & & & 4 & -1 & \\ & & & & -1 & & -1 & 4 & -1 \\ & & & & & -1 & & -1 & 4 \end{array} \right)$$

In contrast to the previous example $-\Delta v = f$ and thus $AU = F$ we now compute the eigenvalues and eigenvectors of A as we seacrh for solutions of $AU = \lambda U$.

Difference methods for hyperbolic problems

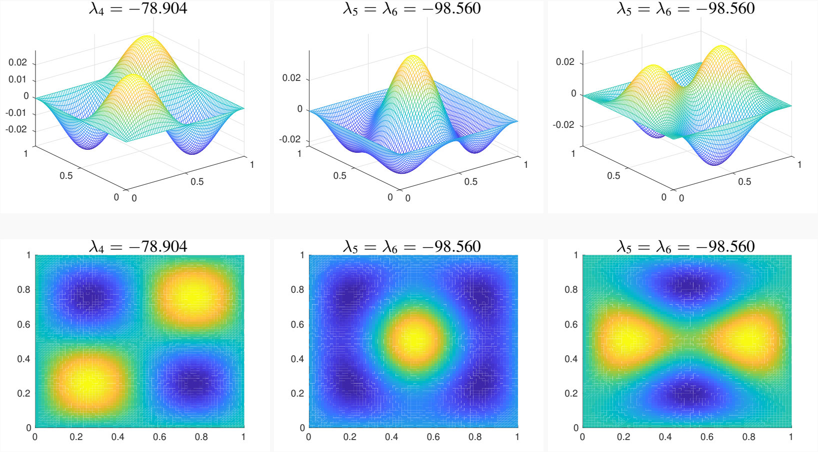
Example (Continuation)

Numerical approximations for $n = 69$ *(eigen value problem for* $\mathbb{R}^{4761\times 4761}$*)*



Example (Continuation)

Numerical approximations for $n = 69$ *(eigen value problem for* $\mathbb{R}^{4761 \times 4761}$*)*



Example (Continuation)

Computing the complete solution based on the eigenmodes. The single eigenmodes behave as

$$u_j(x,t) = \left(a_j \cos\left(\sqrt{\lambda_j} t\right) + b_j \sin\left(\sqrt{\lambda_j} t\right)\right) V_{:,j}, \quad j = 1, 2, 3, \ldots.$$

The complete solution reads

$$u(x,t) = \sum_{j=0}^{n} \left(a_j \cos\left(\sqrt{\lambda_j} t\right) + b_j \sin\left(\sqrt{\lambda_j} t\right)\right) V_{:,j}$$

with a number of $n \leq N$ eigenmodes. The coefficients a_j depend on the initial data and the coefficients b_j on the initial speed.

Difference methods for hyperbolic problems

Procedure for an arbitrary initial displacement and without initial speed:

1. compute the eigenmodes $v_i \in \mathbb{R}^N$, store them in $V = (v_1, \ldots, v_N)$
2. decompose the discretized initial displacement into the eigenmodes
$$U(:,0) = Va$$
with coefficients $a \in \mathbb{R}^N$
3. due to $w''(t) = -\lambda w(t)$ the oscillation for mode λ_i is
$$w_i(t) = \alpha \cos(\sqrt{\lambda_i} t) + \beta \sin(\sqrt{\lambda_i} t)$$
and as $w'(x,0) = 0$ and all $v_1, \ldots, v_N$ are linear independent it holds
$$\beta_1 = \ldots = \beta_n = 0, \quad \alpha_i = a_i$$
4. the numerical solution is
$$U_i^j = \sum_{\ell=1}^{n} a_\ell \cos\left(\sqrt{\lambda_\ell t_j}\right) V_{\ell,j}$$
based on the first $n \leq N$ eigenmodes

Example

We solve the 2D wave equation $u_{tt} = \Delta u$ *for* $\Omega = (-1,1)^2$ *and with the initial condition*

$$\begin{aligned} u(x,y,0) &= u_0(x,y) = \exp(-200(x^2+y^2)),\\ u_t(x,y,0) &= 0 \end{aligned}$$

and on the boundary holds $u(x,y,t) = 0$ *for all* $t \geq 0$.

BUT: for $h = 0.02$ *and thus* $A \in \mathbb{R}^{9801\times 9801}$ *we have to use a large number of eigenmodes to get reasonable results. For* $m = 600$ *the results are*