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13. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 13.1:
Consider the problem

$$\begin{aligned} -u''(x) + u(x) &= x^2 \sin(\pi x) \quad \text{for } x \in \Omega = (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

(a) Compute the weak formulation of the PDE.

(b) Apply the Rayleigh-Ritz method for the basis functions $\varphi_1(x) = x(x-1)$, $\varphi_2(x) = \sin(\pi x)$.

(c) Apply the Rayleigh-Ritz method using a discretization with $h = 0.01$ and hat functions as basis.

Problem 13.2:
Derive a weak formulation of the boundary value problem

$$-xu_{xx} - u_{yy} + u_{xy} + 2u_x + u = 1 \quad \text{on } \Omega = (0,1)^2,\ u|_\Omega = 0.$$

Problem 13.3:

(a) Let T_0 be the reference triangle with the vertices $(0,0)^T$, $(1,0)^T$ and $(0,1)^T$. Compute the integral

$$\int_{T_0} x^2 y^3 \,\mathrm{d}x\,\mathrm{d}y.$$

(b) Compute the affin linear transformation from a general triangle T with the vertices $(x_1, y_1)^T$, $(x_2, y_2)^T$, $(x_3, y_3)^T$ to the reference triangle.

(c) Use a cubature formula of degree 2 from the lecture and approximate

$$\int_T \sin(x)\sin(y)\,\mathrm{d}x\,\mathrm{d}y$$

for the triangle T with the vertices $(1,0)^T$, $(3,1)^T$, $(0,3)^T$.

Problem 13.4:
Consider the boundary value problem

$$-u''(x) = f(x), \quad x \in (0,\pi), \quad u(0) = u(\pi) = 0$$

on an equidistant grid $x_j = jh$, $j = 1, \ldots, n-1$, $h = \pi/n$. Applying finite differences for $n = 40$ results in the linear system of equations $A_h u_h = f_h$ with $A_h = h^{-2}\text{tridiag}(-1, 2, -1)$ and $f_h = (f(x_1), \ldots, f(x_{n-1}))^T$.

Implement a two grid method using the damped Jacobi method ($\omega = 2/3$) to solve the linear system of equations $A_h u_h = f_h$ with $f(x) = \sin(3x)$ for $n = 40$ as follows:

Let x_0 be the starting vector. The iteration reads (e.g. for $i = 0, \ldots, 10$):

Restrict the residual of the fine grid

$$r_h = f_h - A_h x_i$$

to the coarser grid with $2h$ by determining r_{2h} using the associated entries in r_h. Solve the equation for the error

$$A_{2h} d_{2h} = r_{2h}$$

exactly. Prolong the defect d_{2h} to the fine grid with the step size h by linear interpolation in the form $d_h := P d_{2h}$ with P being the prolongation operator. The current iteration is then corrected for the prolonged defect

$$\tilde{x}_{i+1} := x_i + d_h.$$

Starting from $\tilde{x}_{i+1}$ on this grid, carry out 10 iterations of the damped Jacobi method with $\omega = 2/3$. This is the new iteration x_{i+1}.