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Exam „Analysis and Numerics of Partial Differential Equations“

Problem 1 (5 points):

(a) Find the integral curves of $\vec{v}(x, y, z) = \left(x, x^2/y, xz\right)^T$.

(b) Give the general solution of $xu_x + x^2/yu_y + xzu_z = 0$ for $u : \mathbb{R}^3 \to \mathbb{R}$.

(c) Solve the initial value problem $xz_x + x^2/yz_y = xz$ with $z(0, y) = 1 + y^2$ for $z : \mathbb{R}^2 \to \mathbb{R}$.

(d) Consider the PDE from (c) but with the initial condition $z(t, t) = \exp(t)$ for $t \in \mathbb{R}$. Check whether there exists no solution, a unique solution or infinitely many solutions and justify your answer.

Problem 2 (5 points):

Write down the Lax-Friedrich scheme to approximate the solution of

$$\begin{aligned} u_t + 0.5(t+1)uu_x &= 0 \quad \text{for } x \in \Omega = (0, 5),\ t > 0, \\ u(x, 0) &= \max\left(0, 1 - (x-2)^2\right) \quad \text{for } x \in \Omega \end{aligned}$$

using $h = 0.5$. Use the scheme as well as $k = 0.5$ and calculate the approximation for $u(2.5, 1)$. What is the maximal time step size to calculate approximation for $t \in [0, 1]$ in a stable way?

Problem 3 (4 points):

Transform the PDE

$$u_{xx} + 2u_{xz} + 2u_{yy} + 2.5u_{xy} + u_z = 0$$

for $u(x, y, z)$ into canonical form. To this end calculate a proper coordinate transformation. Write down the transformed PDE.

Problem 4 (5 points):

Consider the domain $\Omega = (0, 3)^2 \setminus [1, 2]^2$ and the initial boundary value problem

$$\begin{aligned} -\Delta u &= 2 \quad \text{on } \Omega, \\ u(x, y) &= |x - 1.5| + |y - 1.5| \quad \text{on } \partial\Omega \end{aligned}$$

(a) Make a sketch of the domain and the grid using $h_x = h_y = 0.5$ and number the grid points.

(b) Give the matrix of the associated linear system of equations to approximate $u(x, y)$ at the grid points using finite differences. Is the matrix irreducible?

(c) Compute the right hand side of the system of equations for the equations associated with the three grid points

$$(0.5, 0.5), \quad (1, 0.5), \quad (2.5, 1.5).$$

Problem 5 (6 points):

Calculate the solution $u(x,t)$ of the following initial value problem analytically in form of an infinte sum (Fourier)

$$
\begin{aligned}
&u_{tt} - u_{xx} = 0, && 0 < x < 1, t > 0,\\
&u(0,t) = u(1,t) = 0, && t \geq 0,\\
&u(x,0) = 1 - 2|x - \frac{1}{2}|,\ u_t(x,0) = 0, && 0 < x < 1.
\end{aligned}
$$

HINT: Reformulate the integration to use $2x$ instead of $1 - 2|x - \frac{1}{2}|$.

Problem 6 (5 points):

Consider the parabolic PDE, the boundary and the initial conditions

$$
\begin{aligned}
&u_t(x,y,t) = \Delta u(x,y,t) && \text{in } \Omega = (0,1)^2,\ t > 0\\
&u(x,y,t) = 0 && \text{for } (x,y) \in \partial\Omega,\ t \geq 0\\
&u(x,y,0) = \sin(\pi x)\sin(\pi y) && \text{for } (x,y) \in \bar{\Omega}.
\end{aligned}
$$

Transform the PDE into an ODE-system using $h_x = h_y = 0.25$ in terms of the method of lines. Give also the initial values for the ODE-system.

Write down the schemes for Euler explicit, Euler implicit and Crank-Nicholson for arbitrary time step size $k > 0$. Give formulas for Y_{i+1} in matrix-vector notation for all three schemes.