University of Rostock
Institute of Mathematics
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Exam „Analysis and Numerics of Partial Differential Equations“

Problem 1 (5 points):

Consider the vector field

$$\overrightarrow{v}(x,y,z) = \left(yz, xz, xyz^2\right)^T.$$

(a) Find the integral curves of $\overrightarrow{v}(x,y,z)$.

(b) Compute the general solution of $yzu_x + xzu_y + xyz^2u_z = 0$ for $u : \mathbb{R}^3 \to \mathbb{R}$.

(c) Solve the initial value problem $yzz_x + xzz_y = xyz^2$ with $z(0,y) = y^2$ for $z : \mathbb{R}^2 \to \mathbb{R}$.

(d) Consider the PDE from (c) but with the initial condition $z(t,t) = t$ for $t \in \mathbb{R}$. Check whether there exists no solution, a unique solution or infinitely many solutions and justify your answer.

Problem 2 (5 points):

Consider the initial value problem

$$u_t - \cos(t\pi)u_x = 0 \quad \text{for } x \in \Omega = (-3,3),\ t > 0,$$
$$u(x,0) = \frac{1}{x^2+1} \quad \text{for } x \in \Omega.$$

(a) Apply the Lax-Friedrich scheme with $h = 0.5$ and $k = 0.5$ and approximate $u(1,1)$.

(b) Apply the Lax-Wendroff scheme with $h = 1$ and $k = 1$ and approximate $u(1,1)$.

(c) What is the maximal stepsize for the Lax-Friedrich scheme with $h = 0.5$ to get stable approximations?

Problem 3 (5 points):

Determine the type of the PDE

$$u_{xx} + 3u_{xy} + 2u_{yy} + u_x = 0$$

for $u(x,y)$. Transform the PDE into canonical form with $u_{\xi\xi} = 0$ and $u_{\eta\eta} = 0$ and write down the transformed PDE.

Please turn the page.

Problem 4 (5 points):

Consider the boundary value problem

$$\begin{aligned} \Delta u(x,y) &= \sqrt{x^2+y^2}-1 && \text{in } \Omega = \{(x,y)\in\mathbb{R}^2 :\ x^2+y^2<1\}, \\ u(x,y) &= x^2+y^2-1 && \text{for } (x,y)\in\bar{\Omega}. \end{aligned}$$

(a) Use the new coordinates $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$ as well as

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}$$

and transform the PDE and the boundary condition into polar coordinates.

(b) Apply a discretization with the stepsizes $h_r = 0.5$ in r-direction and $h_\varphi = 0.25\pi$ in φ-direction. Make a sketch and label all inner points.

(c) Determine the matrix of the associated linear system of equations to approximate $u(x,y)$ at the grid points using finite differences.

(d) Compute the right hand side of the system of equations for the equations associated with the two grid points $(0,0)$ and $(0.5,0)$.

Problem 5 (5 points):

Solve the boundary value problem

$$\begin{aligned} -\Delta u(x) &= 4 && x\in\Omega = \{x\in\mathbb{R}^2 :\ \|x\|_2^2 < 3\} \\ u(x) &= v(x) && x\in\partial\Omega \end{aligned}$$

with $v(x) = 1 + 2x_1 + 2x_2 + 3x_1^2 + 3x_1x_2 + 3x_2^2$. To this end

(a) compute $\Delta v(x)$ and

(b) construct the solution $u(x)$ as a sum $u(x) = v(x) + w(x)$ with a $w(x)$ that fulfills $\Delta w(x) = -4 - \Delta v(x)$ in Ω and $w(x) = 0$ on the boundary of Ω.

Problem 6 (5 points):

Let $\Omega = (0,2)$. Consider the initial-boundary value problem

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) && \text{for } (x,t)\in\Omega\times[0,\infty), \\ u(x,0) &= \max(1-2|x-1|,0) && \text{for } x\in\bar{\Omega}, \\ u(x,t) &= 0 && \text{for } x\in\partial\Omega,\ t\geq 0. \end{aligned}$$

Apply the method of lines with $h = 0.25$ and $k = 0.25$.

(a) Transform the PDE into an ODE system.

(b) Write down the iteration for Heun's method in matrix-vector notation and compute an approximation for $u(1, 0.25)$.

(c) Write down the iteration for Crank-Nicolson in matrix-vector notation.