University of Rostock
Institute of Mathematics
PD Dr. M. Sawall

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# Exam „Analysis and Numerics of Partial Differential Equations“

## Problem 1 (5 points):

(a) Find the integral curves of

$$\vec{v}(x,y,z) = \begin{pmatrix} \exp(x) \\ x\sqrt{y+1} \\ 2xz\exp(x) \end{pmatrix}.$$

(b) Give the general solution $u : \mathbb{R}^3 \to \mathbb{R}$ of

$$\exp(x)u_x + x\sqrt{y+1}u_y + 2xz\exp(x)u_z = 0.$$

(c) Solve for $z : \mathbb{R}^2 \to \mathbb{R}$ the initial value problem

$$\exp(x)z_x + x\sqrt{y+1}z_y = 2xz\exp(x), \qquad z(0,y) = 4(y+1).$$

## Problem 2 (5 points):

Consider the problem

$$\begin{aligned} &u_t + \sqrt{t}u_x = 0 \quad \text{for } x \in \Omega = (-3,3),\ t > 0, \\ &u(x,0) = \max\left(0, 1-(x+1)^2\right) \quad \text{for } x \in \Omega. \end{aligned}$$

(a) Give the Lax-Wendroff scheme for this problem using $h = 0.5$ and $k = 0.25$.

(b) Apply the scheme from (a) and compute approximations to $u(-1,2k)$, $u(-0.5,2k)$ and $u(0,2k)$. Use the fact that the approximations for the first time layer are equal to the initial condition, thus $U_{:,1} = U_{:,0}$.

(c) Up to which $t$ are the approximations with this scheme numerically stable?

## Problem 3 (6 points):

Consider $\Omega = (0,1)^2$ and the second order PDE

$$2u_{xx} + 3u_{xy} + u_{yy} - 7u_x - 2u_y = 2.$$

(a) Classify the PDE as elliptic, parabolic or hyperbolic.

(b) Compute a coordinate transformation and transform the PDE into canonical form.

(c) Compute a weak formulation of the PDE.

Please turn the page.

Problem 4 (6 points):

Consider the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : \ |x| + |y| < 1\}$ and the initial boundary value problem

$$\Delta u = 0 \quad \text{on } \Omega,$$
$$u(x, y) = x^2 + 2y \quad \text{on } \partial\Omega.$$

(a) Make a sketch of the domain and discretize it by a grid with $h_x = h_y = 1/3$. Number the grid points.

(b) Write down the equations for the grid points

$$(-\frac{2}{3}, 0), \quad (-\frac{1}{3}, 0), \quad (-\frac{1}{3}, \frac{1}{3}).$$

(c) What is the maximum of the analytical solution $u(x, y)$?

(d) Draw a triangulation of the domain with at least 5 inner points and no hanging node.

Problem 5 (5 points):

Compute the analytical solution of the initial-boundary value problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \ t > 0, \\ &u(0, t) = u(1, t) = 0, \quad t \geq 0, \\ &u(x, 0) = \min(2x, 2 - 2x), \ u_t(x, 0) = 0. \end{aligned}$$

To this end compute the coefficients for the infinite sum. It is not necessary to write down the infinite sum finally, the coefficients are enough.

Problem 6 (3 points):

Consider $\Omega = (0, 1)$ and the problem

$$\begin{aligned} &u_t(x, t) = u_{xx}(x, t) + \sin(x\pi) && \text{for } x \in \Omega, \ t > 0, \\ &u(x, t) = 0 && \text{for } x \in \partial\Omega, \ t \geq 0, \\ &u(x, 0) = (1 - 2|x - 0.5|) && \text{for } x \in \bar{\Omega}. \end{aligned}$$

Apply the method of lines with $h = 0.25$ and specify the ODE system. Give the formulas for Euler explicit and Crank-Nicholson for this problem in matrix-vector notation using the time step size $k$.