University of Rostock
Institute of Mathematics
PD Dr. M. Sawall

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Exam „Introduction to Numerical Mathematics“

Problem 1 (5 points):

Compute approximations to the linear system of linear equations $Ax = b$ with

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

(a) Calculate two steps using Jacobi's method starting with $x^{(0)} = (1,1,1)^T$.

(b) Does Jacobi's iteration converge to the correct solution for any starting vector?

(c) Give the error propagation matrix of Jacobi's method for this certain problem.

(d) How many steps are necessary to get an approximation x_k with $\|x_k - x^*\| < 10^{-3}$ with x^* being the true solution? Apply an a-priori estimation and use that the 2-norm of the error propagation matrix is $\sqrt{0.5}$.

Problem 2 (5 points):

The matrix

$$A = \begin{pmatrix} 2.25 & 1.5 & -1.5 \\ 1.5 & 2 & 0 \\ -1.5 & 0 & 6 \end{pmatrix}$$

is real and symmetric and has thereby only real eigenvalues.

(a) Use Gerschgorin-circles to give intervals for the eigenvalues. Notice that A is symmetric.

(b) Compute one step with the power method for the starting vector $x = (0,0,1)^T$ and give the Rayleigh-coefficient.

(c) Compute one step with the inverse power method for the starting vector $x = (\sqrt{0.5}, -\sqrt{0.5}, 0)^T$ and give the Rayleigh-coefficient.

To solve the system of linear equations you can use the Cholesky decomposition of A that reads

$$A = C^T C, \quad C = \begin{pmatrix} 1.5 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

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Problem 3 (5 points):

The functions

$$f(x) = x^3, \quad g(x) = \exp(-x)$$

intersect in exact one point x^*.

(a) Formulate a fixed-point form $\varphi(x)$ to approximate x^*.

(b) Give an interval I so that the conditions for Banach's fixed point theorem are fullfilled for I and φ.

(c) Select a proper starting value and compute 3 iterations.

(d) How many steps are necessary to get an approximation with $|x_k - x^*| < 10^{-3}$ for sure?

(e) Formulate the Newton iteration to compute x^* and compute 2 steps for $x_0 = 1$.

Problem 4 (4 points):

Consider the points

$$(x_0, y_0) = (-1, 2), \quad (x_1, y_1) = (0, 1), \quad (x_2, y_2) = (1, 1).$$

(a) Compute the interpolation polynomial $p(x)$ for these points in Lagrange-presentation.

(b) Compute the interpolation polynomial that interpolates also $(x_3, y_3) = (2, -1)$ using Newton's scheme.

(c) Compute the Hermite interpolation polynomial using the additional request $p'(2) = -2$.

Problem 5 (6 points):

Given is the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{x^2 + 1}.$$

(a) Compute an approximation to $f''(0)$ using central differences of 2nd order for $h = 0.01$.

(b) Use composite Simpson's rule S_4 with $n = 4$ segments to approximate

$$I = \int_{-5}^{5} f(x)\, \mathrm{d}x.$$

(c) Use composite Gaussian rule Q_1 with 2 nodes and 2 intervals to approximate I.

Use symmetry.

(d) Apply the coordinate transformation $x(z) = \tan(z)$ with

$$x' = \frac{\mathrm{d}x}{\mathrm{d}z} = \frac{1}{\cos(z)^2} \quad \Rightarrow \quad \mathrm{d}x = \frac{1}{\cos(x)^2}\, \mathrm{d}z$$

and composite Gaussian rule Q_1 with 2 nodes and 2 intervals to approximate

$$\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x.$$

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Problem 6 (5 points):

Consider the initial value problem

$$\begin{aligned} y'(x) &= 1 + (y(x))^2, \quad x \in [0, 0.5], \\ y(0) &= 0. \end{aligned}$$

(a) Apply two steps of Euler's method using $h = 0.25$.

(b) Apply one step using $h = 0.5$ and the explicit Runge-Kutta method defined by

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array} .$$

(c) Apply one step of the implicit Euler method using $h = 0.5$.

(d) Compute the order of the method from task (b).