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Suggested solutions for self-study & additional practice for the 4. Tutorial

Sample solution for the additional exercise 15:

The first two iterations for Jacobi's method:

$$x^{(1)} = \begin{pmatrix} 0.25 \\ 0.25 \\ 0 \\ 0 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} -0.1875 \\ -0.1875 \\ -0.1875 \\ 0.1875 \end{pmatrix}.$$

The first two iterations for Gauss-Seidel's method:

$$x^{(1)} = \begin{pmatrix} 0.25 \\ 0.0625 \\ -0.234375 \\ 0.1875 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} -0.1875 \\ -0.35546875 \\ -0.3388671875 \\ 0.296875 \end{pmatrix}.$$

Sample solution for the additional exercise 16:

For Jacobi's method we compute

$$(I - D^{-1}A) = \begin{pmatrix} 0 & 0.25 & 0 & 0.25 \\ 0.25 & 0 & 0.25 & 0 \\ 0 & 0.25 & 0 & 0 \\ -0.25 & 0 & 0 & 0 \end{pmatrix}.$$

If we choose the matrix norm induced by the vector norm $||\cdot||_2$, we get

$$||(I - D^{-1}A)||_2 = 0.405 < 1.$$

For Gauss-Seidel's method we obtain

$$(I - (L + D)^{-1}A) = \begin{pmatrix} 0 & 0.25 & 0 & 0.25 \\ 0 & 0.0625 & 0.25 & 0.0625 \\ 0 & 0.015625 & 0.0625 & 0.015625 \\ 0 & -0.0625 & 0 & -0.0625 \end{pmatrix}$$

and

$$||(I - (L + D)^{-1}A)||_2 = 0.385 < 1.$$

Hence for both methods each starting vector converges to the solution of $Ax = b$. Alternatively we can use any other matrix norm induced by vector p-norms.

Sample solution for the additional exercise 17:

The a-priori estimate is given by

$$||x^{(k)} - x^*|| \leq \frac{q^k}{1-q} \left|\left|x^{(1)} - x^{(0)}\right|\right|,$$

where $q = ||I - B^{-1}A||$. Using the the Euclidean norm and our results from above, we get the inequalities

$$\text{Jacobi's method:} \qquad \frac{0.405^k}{1-0.405} \cdot 1.0607 \leq 10^{-5}$$

$$\text{Gauss-Seidel's method:} \qquad \frac{0.385^k}{1-0.385} \cdot 1.904 \leq 10^{-5}$$

Hence we need at least $k = 14$ steps to achieve the given accuracy in both cases.