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Suggested solutions for self-study & additional practice for the 7. Tutorial

Sample solution for the additional exercise 28:

a) We use the function $\varphi(x) = k\sqrt{0.75k^2 + 2k + 1 + x}$. For the exact value we solve

$$s = k\sqrt{0.75k^2 + 2k + 1 + s} \quad \Rightarrow \quad s_1 = \frac{k}{2}(3k+2),\ s_2 = -\frac{k}{2}(k+2).$$

The value s_2 is not of interest any more, thus $s = \frac{k}{2}(3k+2)$.

For the analysis we get

$$\varphi'(x) = \frac{k}{2\sqrt{0.75k^2 + 2k + 1 + x}} = \frac{k}{\sqrt{3k^2 + 8k + 4 + 4x}}$$
$$\varphi'(x_1) = \frac{k}{\sqrt{9k^2 + 12k + 4}} \in [0,1)\ \forall\ k \in \mathbb{N}$$

and we select e. g. the interval $I = [\alpha, \beta]$ with $-\frac{k^2}{2} - 2k - 1 < \alpha < \frac{k}{2}(3k+2)$ and $\beta > \frac{k}{2}(3k+2)$. Thus $|\varphi'(x)| < 1\ \forall\ x \in I$.

b) We have to analyze the iteration $x^{(k+1)} = \varphi(x^{(k)})$ with $\varphi(x) = \cos x$.

The intervall $I = [0,1]$ is closed and it holds $\varphi(I) \subseteq I$ as cos is monotonously falling on I and $\cos(0) = 1 \in I$ as well as $\cos(1) < 0.6 \in I$. Further holds $\max_{x\in I} |-\sin(x)| < 0.85 = L$. So the conditions of Banach's fixed-point theorem are satisfied and there exists an unique fixed-point in I and the iteration converges for all starting values $x_0 \in I$.

Sample solution for the additional exercise 29:

a) Isolating x on the left hand side, we get

$$\ln(16 - x) = \sqrt{\frac{2}{3}x^2 + 4} \qquad \Leftrightarrow \qquad x = 16 - \exp\left(\sqrt{\frac{2}{3}x^2 + 4}\right),$$

hence the iteration is given by

$$\varphi_1(x) = 16 - \exp\left(\sqrt{\frac{2}{3}x^2 + 4}\right).$$

b) Isolating x on the right hand side, we get

$$\ln(16 - x) = \sqrt{\frac{2}{3}x^2 + 4} \qquad \Leftrightarrow \qquad x = \pm\sqrt{\frac{3}{2}\left((\ln(16 - x))^2 - 4\right)}.$$

Since we are looking for an intersection in the interval $[1,7]$, it is sufficient to choose the iteration

$$\varphi_2(x) = \sqrt{\frac{3}{2}\left((\ln(16-x))^2 - 4\right)}.$$

c) We compute the Lipschitz constants for φ_1 and φ_2:

$$|\varphi_1'(x)| = \underbrace{\left|\exp\left(\sqrt{\frac{2}{3}x^2+4}\right)\right|}_{\geq \exp\left(\sqrt{\frac{2}{3}1^2+4}\right)} \cdot \underbrace{\left|\left(\frac{2}{3}x^2+4\right)^{-\frac{1}{2}}\right|}_{\geq\left(\frac{2}{3}7^2+4\right)^{-\frac{1}{2}}} \cdot \underbrace{\left|\frac{4}{3}x\right|}_{\geq\frac{4}{3}} \geq 1.9097 > 1,$$

$$|\varphi_2'(x)| = \underbrace{\left|3\ln(16-x)\right|}_{\leq 3\ln(16-1)} \cdot \underbrace{\left|\frac{1}{16-x}\right|}_{(16-7)^{-1}} \cdot \underbrace{\left|\left(\frac{3}{2}\left((\ln(16-x))^2-4\right)\right)^{-\frac{1}{2}}\right|}_{\leq\left(\frac{3}{2}\left((\ln(16-7))^2-4\right)\right)^{-\frac{1}{2}}} \leq 0.81008 < 1.$$

Since $[1,7]$ is a closed subspace of $\mathbb{R}$ and $\varphi([1,7]) \subset [1,7]$, we are allowed to apply Banach's fixed-point theorem. Hence the second fixed-point iteration converges to $x = 2.095856$, whereas the first one does not.

d) For the iteration φ_2 a possible Lipschitz constant L is given by $L = 0.81008$. The a-priori estimate is

$$|x_i - x^*| \leq \frac{L^i}{1-L}|x_1 - x_0|,$$

hence we get the inequality

$$10^{-5} \leq \frac{0.81008^i}{0.18992}1.033163.$$

We need at least 63 iterations to drop the distance $|x_i - x^*|$ below 10^{-5}.

Sample solution for the additional exercise 30:

a) We get the iteration

$$x_{k+1} = x_k - \frac{x_k^3 - 2x_k + 2}{3x_k^2 - 2}$$

and we obtain the following results

$$x_0 = -2, \quad x_1 = -1.666667, \quad x_2 = -1.864583, \quad x_3 = -1.712489.$$

b) Starting at $x_0 = 1$, the first three iterations are given by

$$x_0 = 1, \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 0.$$

Therefore the sequence is periodical and does not converge to the zero of f.