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4. Tutorial on the lecture „Introduction to Numerical Mathematics“

Problem 15:

(a) Apply Jacobi's method and Gauss-Seidel's method to solve $Ax = b$ with

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Use $x^{(0)} = (1, 1, 1)^T$ as starting vector and compute 2 steps for each method.

(b) Does the iteration converge for each starting vector?

(c) Apply an a-priori estimation of the error in order to determine how many steps are necessary to reach an approximation with $\|x^* - x^{(k)}\|_2 < 10^{-4}$.

(d) Apply Jacobi's method to solve $Ax = b$ with

$$A = \text{tridiag}(1, -2, 1) \in \mathbb{R}^{n\times n}, \quad b = (1, \dots, 1)^T \in \mathbb{R}^n$$

for $n = 101$. Use $\|Ax^{(i)} - b\|_2 \le 10^{-6}$ as stopping criterion and $x^{(0)} = -b$ as initial guess. How many iterations are necessary for the Gauss-Seidel iteration?

Problem 16:

Regard the system of linear equations $Ax = b$ with

$$A = \begin{pmatrix} 1 & 0 & \gamma \\ \alpha & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix} \text{ mit } \alpha, \beta, \gamma \in \mathbb{R}, \quad b \in \mathbb{R}^3.$$

Under which conditions at α, β and γ does the Gauss-Seidel method converge for this A and all starting vectors? Determine the error propagation matrices F_J and F_{GS} and their characteristic polynomials $p_J(\lambda)$ and $p_{GS}(\lambda)$. Calculate the zeros of $p_J(\lambda)$ and estimate the magnitudes of the zeros of $p_{GS}(\lambda)$ and use a theorem from the lecture.

Show that for this A, the Gauss-Seidel method converges for all starting vectors if and only if Jacobi's method converges for all starting vectors.

Problem 17:

Let $n \in \mathbb{N}$. The Hilbert-matrix

$$A = (a_{ij})_{i,j=1,\dots,n} \quad \text{with} \quad a_{ij} = \frac{1}{i+j-1}, \ 1 \le i, j \le n$$

is regular bad bad conditioned. The system of linear equation $Ax = b$ with $b_i = \sum_{j=1}^n a_{ij}$ has the solution $x = (1, \dots, 1)^T$.

Use the Gaussian-elimination to solve the system. Up to which n the algorithm returns reasonable results?

Problem 18:

Apply under and over relaxation to solve the system of linear equations from problem 15. Use $\omega \in \{0.5,\ 2/3,\ 1,\ 1.5\}$ and count the steps for Jacobi's method and for Gauss-Seidel iteration.